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FINAL REPORT

THE DYNAMIC ANALYSIS OF EMPTY AND PARTIALLY FULL CYLINDRICAL TANKS

PART I – FREQUENCIES AND MODES OF FREE VIBRATION AND TRANSIENT RESPONSE BY MODE ANALYSIS

by

MELVIN L. BARON and HANS H. BLEICH

Defense Atomic Support Agency
Contract DA-29-044-XZ-557
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Dr. Mario G. Salvadori (Consultant)

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The Dynamic Analysis of Empty and Partially Full Cylindrical Tanks

PART I. Frequencies and Modes of Free Vibrations and Transient Response by Mode Analysis

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List of Symbols [1]

r, θ, z	Cylindrical coordinates, See Fig. (I-1)
$u(z, \theta, t)$, $v(z, \theta, t)$ $w(z, \theta, t)$	Longitudinal, tangential and radial components of the shell displacement. (w is measured positive inward), See Fig. (I-1)
$u(z, \theta)$, $v(z, \theta)$, $w(z, \theta)$	Space dependent parts of u, v, and w in a principal mode.
a	Radius of shell
a_{nj}, $j = 0\text{-}8$	Fourier Series coefficient - See Eq. (VII-18)-(VII-21)
A_{ni}	Generalized coordinate - sloshing modes of fluid filling in shell.
A_R	Cross sectional area of wind girder.
C_n	Normalization Coefficient
C_{ni}	Expansion coefficient for velocity potential ϕ_2.
$\bar{C}_n$	Coefficient - See Eq. (IV-52)-(IV-53).
C_{u_1}, C_{v_1}, C_{w_1}	Constants - See Eq. (I-1)-(I-3).
E	Modulus of Elasticity
f_{ni}	Integral, See Eq. (VII-15).
G_n	Coefficient - See Eq. (IV-95).
h	Shell Thickness
I_{nip}	Integral, See Eq. (VII-32)
I_R	Moment of Inertia of wind girder.
J_n	Bessel Function of first kind of order n.
$k = \dfrac{h^2}{12a^2}$	Coefficient.
K_n	Mass ratio, $K_n = \dfrac{m_{vn}}{m_1} = \dfrac{\epsilon_n \rho a}{m_1}$

(1) Additional symbols in the text are defined as they occur.

L	Height of shell .
$\bar{L}$	Height of wind girder above bottom of tank.
M, M_{ni}	Frequency number, See Eq. (I-22).
$\bar{M}_o$	Generalized mass of shell in mode n = 0, See Eq. (IV-79).
$\bar{M}_1$	Total mass of empty tank = $2\pi a L m_1$.
M_{nm}, M_{nI}	Frequency numbers - See Section (III).
m_1	Mass per unit of area of the empty shell.
m_{vn}	Virtual mass per unit area of the fluid in the n^{th} mode, See Eq. (IV-55).
n	Number of circumferential waves of a mode of vibration.
p	Pressure in fluid.
$P(\theta, z, t)$	Radial blast pressure on shell.
$P(r, \theta, t)$	Vertical blast pressure on fluid surface or on roof of shell
P_{ih}	Radial blast pressure, in subarea "ih". See Section (VII-a)
P_{kp}	Vertical blast pressure on subarea S_{kp}.
$q_n(t)$	Generalized coordinate of shell displacements.
Q_n, Q_{n1}, Q_{n2}	Generalized forces in the n^{th} mode.
S_k	Sector of shell, See Section (VII-a).
S_{kp}	Subarea over which vertical blast forces on the fluid surface of the shell are measured.
t	Time
T	Kinetic energy
$\bar{T}$	See Eq. (V-39)
T_f	Kinetic energy of fluid

U, V, W, X, Y	Coefficients defining the shapes of modes of free vibration of the shell.
V_1 , V_2	Potential Energy of Shell.
v_r, v_θ, v_z	Fluid velocities in the r, θ and z directions.
α_{ni}	Root of transcendental Eq. (IV-23).
β_{ni} , β_{oi}	Coefficient - See Eq. (IV-62) and (IV-93).
$\overline{\beta}_{ni}$	Coefficient - See Eq. (V-112)
δ_n	Conversion Coefficient - See Eq. (V-42)-(V-45).
γ	Fraction denoting the height of the fluid filling in the shell. Height of fluid = γL.
ϵ_n	Coefficient - See Eq. (IV-58).
$\epsilon_{\theta\theta}$, ϵ_{zz}, $\epsilon_{z\theta}$	Shell Strains.
η_n , $\dot{\eta}_{ni}$	Displacement of Fluid Surface.
ν	Poisson's Ratio.
$\xi = L/a$	Ratio of height of tank to radius of tank.
ρ	Mass density of fluid.
ρ_i	Mass density of empty shell.
$\sigma_{\theta\theta}$, σ_{zz} , $\sigma_{z\theta}$	Shell Stresses.
$\varphi(r, \theta, z, t)$	Velocity potential function of fluid
ψ_{nk}	See Eq. (VII-33).
ω, ω_{nj}	Frequency of vibration of empty or partially full shell.

Note: r, θ, or z used as a subscript for the shell displacements u, v, w, denotes differentiation with respect to the particular variable used.

Dots indicate differentiation with respect to time.

I. Modes and Frequencies of Free Vibrations of Empty Cylindrical Tanks.

Consider the empty fuel tank of Fig. (I-1) consisting of a thin cylindrical shell of height L and radius "a", free at the top and simply supported at the base. The positive directions of the displacement components u, v and w are shown in the figure.

An analysis of the frequencies and modes of such a structure has been made in Reference (I)[1] on the assumption that the shell acts as a membrane with no bending stiffness. In addition, a procedure for obtaining the frequencies of the shell including bending effects is presented in this reference. The condition of simple support at the base of the shell requires in general that the displacements $u(z,\theta,t)$, $v(z,\theta,t)$ and $w(z,\theta,t)$ and the moment M_z be zero at $z = 0$. If the shell is considered to be a membrane with no bending stiffness, only the conditions that $u(z, \theta, t)$ and $v(z, \theta, t)$ are zero at $z = 0$ can be enforced.[2]

The transcendental equations for the frequencies of vibration and the expressions for the mode shapes corresponding to these frequencies, according to Reference (1), are extremely complicated and are very difficult to apply to forced vibration problems involving the response of the

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(1) "Dynamic Response of Cylindrical Tanks" by F. L. DiMaggio, Armed Forces Special Weapons Project, Contract DA-29-004-XZ-54, AFSWP No. 1075, May, 1958.

(2) Any combination of two displacements set equal to zero could suffice to solve the boundary value problem for the membrane shell. However, as shown in Reference (1), the particular combination used gives rise to very small displacements $w(z, \theta, t)$ at the base of the shell. The correction to the displacement curve for $w(z, \theta, t)$ when bending stiffness is included in the analysis and the condition $w(z, \theta, t) = 0$ at $z = 0$ is enforced is shown to be very small under these conditions.

shell to transient blast loadings. In the present section of this report, using these frequencies and shapes as a guide, approximate values for the frequencies and modes of free vibrations of thin cylindrical shells of constant thickness are obtained. These approximate modes are of a relatively simple form and can easily be used in Section VII of this Report in which an analysis of the forced vibration of empty and partially full tanks is presented. Since the application of these results to both steel shells and steel shells with concrete shielding is contemplated, both membrane and bending effects will be considered in the following analysis.

The modes of free vibrations of the shell may be characterized by an integral number n_1 the number of circumferential waves in the mode. For each configuration n_1 there exists an infinite number of frequencies and corresponding mode shapes. In general, for $n \neq 0$, we will only require the lowest one of these frequencies and mode shapes. For the case $n = 0$, special conditions prevail, and this case will be discussed in Part (e) of this section.

The displacements u, v, w of the shell in free vibrations can be expanded into a power series in the coordinate z:(2a)

$$u(z, \theta, t) = \left[C_{u_1} \frac{z}{a} + C_{u_2} \frac{z^2}{a^2} + C_{u_3} \frac{z^3}{a^3} + \dots \right] \cos n\theta \, e^{i\omega t} \qquad \text{(I-1)}$$

$$v(z, \theta, t) = \left[C_{v_1} \frac{z}{a} + C_{v_2} \frac{z^2}{a^2} + C_{v_3} \frac{z^3}{a^3} + \dots \right] \sin n\theta \, e^{i\omega t} \qquad \text{(I-2)}$$

$$w(z, \theta, t) = \left[C_{w_0} + C_{w_1} \frac{z}{a} + C_{w_2} \frac{z^2}{a^2} + \dots \right] \cos n\theta \, e^{i\omega t} \qquad \text{(I-3)}$$

(2a) From the reference of Footnote (1)of this section, it is evident that the z dependent terms of the mode shapes are well behaved functions which can be expanded into power series about the point $z = 0$.

where the constant terms C_{u_o} and C_{v_o} are taken equal to zero due to the boundary conditions at the base of the shell. In the membrane analysis as presented in Reference (1), the radial displacement cannot be zero at $z = 0$ and the constant term C_{w_o} must be included in the series for $w(z, \theta, t)$. The choice of the number of terms in each series which must be retained in an approximate analysis is governed by the simplest combination of terms required to obtain good approximations to the frequencies, membrane strains and membrane stresses of Reference (1). It should be noted that the exact membrane displacements do not differ very much from straight lines and hence, relatively few terms need be retained in Eq. (I-1) - Eq. (I-3). A detailed discussion of the above considerations appears in Section II of this Report.

Two sets of approximate displacements are considered in this section; 1) a set in which five constants are retained in Eq. (I-1) to (I-3) and 2) a set in which three constants are retained. While the latter will give satisfactory estimates of frequencies for most applications, the former are required to give accurate values of strains and stresses for use in forced vibration problems.

(a) Approximation Using Five Constants. - Membrane and Bending Effects $n \neq 0$

Let the displacements of the shell (Fig. 1-1) be given by[2b]

$$u(z, \theta, t) = u(z, \theta)e^{i\omega t} = \left[U \frac{z}{a} + X\left(\frac{z^2}{a^2} - \frac{3Lz}{4a^2}\right)\right] \cos n\theta e^{i\omega t} \tag{I-4}$$

$$v(z, \theta, t) = v(z, \theta)e^{i\omega t} = V \frac{z}{a} \sin n\theta e^{i\omega t} \tag{I-5}$$

$$w(z, \theta, t) = w(z, \theta)e^{i\omega t} = \left[Y + W\left(\frac{z}{a} - \frac{L}{2a}\right)\right] \cos n\theta e^{i\omega t} \tag{I-6}$$

where $n = 1, 2, \ldots.$ is an integral number indicating the number of circumferential waves in the mode, ω is the frequency of free vibrations and U, V, W, X and Y are the five constants.

The Rayleigh-Ritz method[2c] will be used to determine the frequencies and mode shapes of the free vibrations of the shell. To use this method, the kinetic and potential energies of the shell must be determined in terms of the parameters U, V, W, X and Y. The kinetic energy of the shell is given by the relation

$$T = \frac{m_1}{2} \int_0^{2\pi} \int_0^L (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\, a\, d\theta\, dz \tag{I-7}$$

where m_1 is the mass per unit area of the shell

$$m_1 = \rho_1 h \tag{I-8}$$

and dots indicate differentiation with respect to time.

(2b) The expressions in the brackets in Eq. (I-4) and (I-6) for u and w are of the form $(az + bz^2)$ and $(c + dz)$ respectively. The particular

(2b. Cont.). . .combinations of coefficients used in these equations were chosen so that the kinetic energy of the empty shell, Eq. (I-9), would contain only the squares of the coefficients U, V, W, X and Y. This leads to a considerable simplification in the frequency determinant, Eq. (I-23), by making the YW and WY coefficients equal to zero and insuring that all terms containing the frequency number M lie only on the main diagonal.

(2c) The Rayleigh-Ritz Method is based on the condition that the variation of the difference

$$L = T-V$$

vanishes for arbitrary variations of the deflection curve consistent with the boundary conditions, i.e. the geometrical restraints, of the problem. By chosing the displacements to be functions containing the arbitrary coefficients U, V, W, X and Y, the quantities T and V can be expressed in terms of these coefficients, See Eq. (I-9)-(I-14) and the function T-V may be formed. This quantity is now a function of the variables U, V, W, X, and Y. The condition that the variation of T-V vanishes for arbitrary variation in U, V, W, X and Y requires that

$$\frac{\partial(T-V)}{\partial C_1} = 0$$

where C_1 takes on successively the values U, V, W, X and Y. This leads to a system of five simultaneous homogeneous equations on U, V, W, X and Y; for non zero solutions of this system of equations, the determinant of the coefficients must be set equal to zero thus leading to the frequency determinants which appear later in the paper. See, for example, "Vibration Problems in Engineering" by S. Timoshenko, D. VanNostrand and Co., Third Edition, Jan. 1955, Pg. 381 ff. In this reference the method is called just the Ritz method. Also "Mathematical Methods in Engineering" by T.Von Karman and M. Biot, McGraw Hill, 1940, Pg. 352 ff.

Substituting Eq. (I-4) to (I-6) into Eq. (I-7), the kinetic energy becomes

$$T = \frac{m_1 \pi a \omega^2}{6} \quad \frac{L^3}{a^2} \left[U^2 + V^2 + \frac{W^2}{4} + \frac{3a^2}{L^2} Y^2 + \frac{3L^2}{80\ a^2} X^2 \right] \tag{I-9}$$

The potential energy V stored in the shell can be expressed as a function of the displacements $u(z, \theta)$, $v(z, \theta)$ and $w(z, \theta)$

$$V = V_1 + V_2 \tag{I-10}$$

where the first term V_1 is the membrane strain energy[3]:

$$V_1 = \frac{E}{2(1-\nu^2)} \frac{h}{a} \int_0^{2\pi} \int_0^L \left[a^2 u_z^2 + (v_\theta - w)^2 + 2a\nu u_z (v_\theta - w) + \frac{(1-\nu)}{2} (u_\theta + a v_z)^2 \right] dz d\theta \tag{I-11}$$

while the second term V_2 represents the strain energy of bending and coupling terms between the membrane and bending strains:

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(3) "Tables for the Frequencies and Modes of Free Vibration of Infinitely Long Thin Cylindrical Shells" by M. L. Baron and H. H. Bleich, Journal of Applied Mechanics, Vol. 21, No. 2, June 1954. Transactions of the American Society of Mechanical Engineers.

$$V_2 = \frac{E}{24(1-\nu^2)} \frac{h^3}{a^3} \int_0^{2\pi}\!\!\int_0^{L} \left[a^4 w_{zz}^2 + (w_{\theta\theta} + w)^2 + \left(\frac{1-\nu}{2}\right)(a w_{z\theta} - u_\theta)^2 + \frac{3(1-\nu)}{2}\, a^2 (v_z + w_{z\theta})^2 + 2a^2 \nu w_{zz}(w_{\theta\theta} + v_\theta) + 2a^3 u_z w_{zz} \right] dz\,d\theta \quad (4) \qquad \text{(I-12)}$$

The subscripts indicate partial differentiation with respect to z or θ. Substituting the expressions for u, v and w into Eq. (I-11)and Eq. (I-12), the potential energy of the shell is given by:

$$V_1 = \frac{Eh\pi}{2(1-\nu^2)}\, \ell \; \Bigg\{ U^2 \left[1 + \frac{(1-\nu)n^2\ell^2}{6}\right] + V^2 \left[\frac{(1-\nu)}{2} + \frac{n^2\ell^2}{3}\right] + W^2 \frac{\ell^2}{12} + X^2 \left[\frac{19}{48}\ell^2 + \frac{(1-\nu)n^2\ell^4}{160}\right] + Y^2 + UV \left[\left(\frac{3\nu-1}{2}\right) n\ell\right] + UX \frac{\ell}{2} - UY2\nu - VW \frac{n\ell^2}{6} - VYn\ell + VX \left[\frac{(1+13\nu)}{24} n\ell^2\right] - WX \frac{\nu\ell^2}{3} - XY \frac{\nu\ell}{2} \Bigg\} \qquad \text{(I-13)}$$

---

(4) It is proper to call attention to the fact that the expression for the strain energy V_2 used may not be rigorously correct in the light of recent advanced shell theories[5]. However, the authors are satisfied that the effect of any corrected strain-energy expression in the range of application of these results is insignificant.

(5) "The New Approach to Shell Theory", By E. H. Kennard, Journal of Applied Mechanics, Vol. 75, 1953. Transactions of the American Society of Mechanical Engineers , pp 33-40.

and

$$V_2 = \frac{Eh\pi}{24(1-\nu^2)} \quad \frac{h^2}{a^2} \left\{ \begin{aligned} & W^2 \left[\frac{(1-n^2)^2 \xi^3}{12} + 2(1-\nu)n^2\xi \right] + Y^2 \left[(1-n^2)^2 \xi \right] \\ & + U^2 \left[\left(\frac{1-\nu}{6}\right) n^2 \xi^3 \right] + V^2 \left[\frac{3(1-\nu)\xi}{2} \right] \\ & + X^2 \left[\frac{(1-\nu)n^2\xi^5}{160} \right] - WU \left[\left(\frac{1-\nu}{2}\right) n^2 \xi^2 \right] \\ & + WX \left[\frac{(1-\nu)n^2\xi^3}{24} \right] - WV \left[3(1-\nu)n\xi \right] \end{aligned} \right\} \qquad \text{(I-14)}$$

where $\xi = L/a$.

The Rayleigh-Ritz method for the determination of the frequencies ω and the ratio of the shapes requires that

$$\frac{\partial}{\partial C_i} \left(T - V_1 - V_2 \right) = 0 \qquad \text{(I-15)}$$

where C_i successively takes the values U, V, W, X and Y. Eq. (I-15) leads to the following set of five homogeneous linear equations in the five unknowns U, V, W, X and Y:

$$\left[2M - \frac{6}{\xi^2} - (1-\nu)n^2 - k(1-\nu)n^2 \right] U + \left[-\frac{3n(3\nu-1)}{2\xi} \right] V + \left[k\, \frac{3(1-\nu)n^2}{2\xi} \right] W + + \left[\frac{-3}{2\xi} \right] X + \left[\frac{6\nu}{\xi^2} \right] Y = 0 \qquad \text{(I-16)}$$

$$\left[-\frac{3n(3\nu-1)}{2\xi} \right] U + \left[2M - \frac{3(1-\nu)}{\xi^2} - 2n^2 - k\frac{9(1-\nu)}{\xi^2} \right] V + \left[\frac{n}{2} + k\frac{9n(1-\nu)}{\xi^2} \right] W + \left[-\frac{n}{8}(1+13\nu) \right] X + \left[\frac{3n}{\xi} \right] Y = 0 \qquad \text{(I-17)}$$

$$\left[k\,\frac{3(1-\nu)n^2}{2\ell}\right] U + \left[\frac{n}{2} + \frac{k9(1-\nu)n}{\ell^2}\right] V + \left[\frac{M}{2} - \frac{1}{2} - k\left\{\frac{(1-n^2)^2}{2} + \frac{12(1-\nu)n^2}{\ell^2}\right\}\right] W$$

$$+ \left[\nu - k\left(\frac{1-\nu}{8}\right)n^2\right] X + \left[0\right] Y = 0 \qquad \text{(I-18)}$$

$$\left[-\frac{3}{2\ell}\right] U + \left[-\frac{n}{8}(1+13\nu)\right] V + \left[\nu - k\left(\frac{1-\nu}{8}\right)n^2\right] W$$

$$+ \left[\frac{3}{40}\ell^2 M - \frac{19}{8} - \frac{3(1-\nu)n^2\ell^2}{80} - k\,\frac{3(1-\nu)n^2\ell^2}{80}\right] X + \left[\frac{3\nu}{2\ell}\right] Y = 0 \qquad \text{(I-19)}$$

$$\left[\frac{6\nu}{\ell^2}\right] U + \left[\frac{3n}{\ell}\right] V + \left[0\right] W + \left[\frac{3\nu}{2\ell}\right] X + \left[\frac{6}{\ell^2}\left\{M - 1 - k(1-n^2)^2\right\}\right] Y = 0 \qquad \text{(I-20)}$$

where

$$k = \frac{h^2}{12a^2} \qquad \text{(I-21)}$$

and

$$M = \frac{m_1 a^2 \omega^2 (1-\nu^2)}{Eh} \qquad \text{(I-22)}$$

Nonvanishing solutions of Eq. (I-17)-(I-20) and free vibrations exist only if the determinant of the system vanishes. This leads to the determinantal frequency equation, shown on Page 13.

For given values of ℓ and n, Eq. (I-23) yields five positive roots M_j, defining five mutually orthogonal modes, of frequencies

$$\omega_j^2 = \frac{M_j Eh}{m_1 a^2 (1-\nu^2)} \qquad j = 1, 2, 3, 4, 5 \qquad \text{(I-24)}$$

$$
\begin{vmatrix}
2M-\frac{6}{\xi^2}-(1-\nu)n^2(1+k) & -\frac{3n(3\nu-1)}{2\xi} & \frac{3kn^2(1-\nu)}{2\xi} & -\frac{3}{2\xi} & \frac{6\nu}{\xi^2} \\
-\frac{3n(3\nu-1)}{2\xi} & 2M-2n^2-\frac{3(1-\nu)}{\xi^2}(1+3k) & \frac{n}{2}+\frac{kn9(1-\nu)}{\xi^2} & -\frac{n(1+13\nu)}{8} & \frac{3n}{\xi} \\
\frac{3kn^2(1-\nu)}{2\xi} & \frac{n}{2}+\frac{kn9(1-\nu)}{\xi^2} & \frac{M}{2}-\frac{1}{2}-k\left\{\frac{(1-n^2)^2}{2}+\frac{12(1-\nu)n^2}{\xi^2}\right\} & \nu-\frac{k(1-\nu)n^2}{8} & 0 \\
-\frac{3}{2\xi} & -\frac{n(1+13\nu)}{8} & \nu-\frac{k(1-\nu)n^2}{8} & \frac{3}{40}M\xi^2-\frac{19}{8}-\frac{3(1-\nu)n^2\xi^2}{80}(1+k) & \frac{3\nu}{2\xi} \\
\frac{6\nu}{\xi^2} & \frac{3n}{\xi} & 0 & \frac{3\nu}{2\xi} & \frac{6}{\xi^2}\left\{M-1-k(1-n^2)^2\right\}
\end{vmatrix} = 0
$$

$n \neq 0$

Membrane + Bending Effects
5 Constant Approximation
Empty Shell

(I-23)

The shape of the mode pertaining to a particular frequency ω_j can be found by computing the ratios $\frac{U}{W}$, $\frac{V}{W}$, $\frac{X}{W}$ and $\frac{Y}{W}$ from any four of the Eq. (I-16)-(I-20). In general, the lowest frequency ω_1 only, will be required and for simplicity in the following expressions the subscript j for each of the constants U_j, V_j, W_j, X_j and Y_j will be dropped.

Once the frequency ω and the ratios of the constants have been evaluated for a particular value of n, the displacements of the shell may be written in the form:

$$u(z,\theta) = C_n \left[\frac{U}{W}\frac{z}{a} + \frac{X}{W}\left(\frac{z^2}{a^2} - \frac{3Lz}{4a^2}\right) \right] \cos n\theta \tag{I-25}$$

$$v(z, \theta) = C_n \left[\frac{V}{W}\frac{z}{a} \right] \sin n\theta \tag{I-26}$$

$$w(z, \theta) = C_n \left[\frac{Y}{W} + \left(\frac{z}{a} - \frac{L}{2a}\right) \right] \cos n\theta \tag{I-27}$$

It is convenient to define "normalized modes of vibration" by chosing the constant C_n so that

$$m_i \int_0^L \int_0^{2\pi} (u^2 + v^2 + w^2) a\,d\theta\,dz = 2\pi a L m_i = M_i \tag{I-28}$$

In this case, the modes are normalized to the total mass of the empty shell, M_i. Substituting Eq. (I-25)-(I-27) into Eq. (I-28), the normalization coefficient C_n becomes

$$C_n = \left\{ \frac{6}{\xi^2} \left[\frac{1}{\left(\frac{U}{W}\right)^2 + \left(\frac{V}{W}\right)^2 + \frac{1}{4} + \frac{3}{\xi^2}\left(\frac{Y}{W}\right)^2 + \frac{3}{80}\xi^2\left(\frac{X}{W}\right)^2} \right] \right\}^{1/2} \tag{I-29}$$

Using Eq. (I-29), Eq. (I-25)-(I-27) give the normalized modes of free vibration of the empty tank.

It is of interest to evaluate the normalized strains and the corresponding normalized stresses in each mode. The strains for the plane stress solution of the boundary value problem for the free vibrations of thin cylindrical shells are given by

$$a\epsilon_{\theta\theta} = v_\theta - w = C_n \left[\frac{L}{2a} + (n\frac{V}{W} - 1)\frac{z}{a} - \frac{Y}{W}\right] \cos n\theta \qquad \text{(I-30)}$$

$$a\epsilon_{zz} = au_z = C_n \left[\frac{U}{W} + (\frac{2z}{a} - \frac{3}{4}\frac{L}{a})\frac{X}{W}\right] \cos n\theta \qquad \text{(I-31)}$$

$$a\epsilon_{z\theta} = \frac{1}{2}\left[av_z + u_\theta\right] = \frac{C_n}{2}\left[\frac{V}{W} - n\frac{z}{a}\frac{U}{W} - n\frac{X}{W}\left(\frac{z^2}{a^2} - \frac{3}{4}\frac{L}{a}\frac{z}{a}\right)\right] \sin n\theta \qquad \text{(I-32)}$$

The normalized stresses may then be obtained from the relations

$$\sigma_{\theta\theta} = \frac{E}{a(1-\nu^2)}\left[a\epsilon_{\theta\theta} + \nu\, a\epsilon_{zz}\right] \qquad \text{(I-33)}$$

$$\sigma_{zz} = \frac{E}{a(1-\nu^2)}\left[a\epsilon_{zz} + \nu\, a\epsilon_{\theta\theta}\right] \qquad \text{(I-34)}$$

$$\sigma_{z\theta} = \frac{E}{a(1+\nu)}\left[a\epsilon_{z\theta}\right] \qquad \text{(I-35)}$$

b) Approximation Using Five Constants - Membrane Effects Only

$n \neq 0$

For the case of thin shells in which the ratio of the thickness to the radius, $\frac{h}{a}$, is very small, the strain energy of bending V_2 becomes extremely small with respect to the membrane strain energy V_1, for modes with a low circumferential wave number "n"(6). In such cases, excellent approximations of frequencies, strains and stresses can be obtained if the shell is considered to act as a membrane with no bending stiffness and V_2 is set equal to zero. The range of h/a and n for which the membrane assumption is permissible is discussed in detail in Section III of this report for several cases of interest.

The frequency equation and the mode shapes for the membrane shell are obtained by setting the value of the constant k equal to zero in Eq. (I-16)-(I-20). The system of homogeneous equations then becomes:

$$\left[2M - \frac{6}{\xi^2} - (1-\nu)n^2\right] U + \left[-\frac{3n(3\nu-1)}{2\xi}\right] V + \left[0\right] W + \left[-\frac{3}{2\xi}\right] X + \left[\frac{6\nu}{\xi^2}\right] Y = 0 \tag{I-36}$$

$$\left[-\frac{3n(3\nu-1)}{2\xi}\right] U + \left[2M - \frac{3(1-\nu)}{\xi^2} - 2n^2\right] V + \left[\frac{n}{2}\right] W + \left[-\frac{n}{8}(1+13\nu)\right] X + \left[\frac{3n}{\xi}\right] Y = 0 \tag{I-37}$$

$$\left[0\right] U + \left[\frac{n}{2}\right] V + \left[\frac{M}{2} - \frac{1}{2}\right] W + \left[\nu\right] X + \left[0\right] Y = 0 \tag{I-38}$$

$$\left[-\frac{3}{2\xi}\right] U + \left[-\frac{n}{8}(1+13\nu)\right] V + \left[\nu\right] W + \left[\frac{3}{40}\xi^2 M - \frac{19}{8} - \frac{3(1-\nu)n^2\xi^2}{80}\right] X + \left[\frac{3\nu}{2\xi}\right] Y = 0 \tag{I-39}$$

(6) It should be noted that for higher modes with very large values of n, the effect of the bending strain energy V_2 becomes prevalent and the frequency in the higher modes is controlled by the inextensional effects.

$$\left[\frac{6\nu}{\xi^2}\right] U + \left[\frac{3n}{\xi}\right] V + \left[0\right] W + \left[\frac{3\nu}{2\xi}\right] X + \left[\frac{6}{\xi^2}\left\{M - 1\right\}\right] Y = 0 \qquad \text{(I-40)}$$

The determinantal frequency equation is obtained by setting the determinant of the above system equal to zero, or by setting $k = 0$ in Eq. (I-23). This determinant is shown on Page 18.

For given values of ξ and n, Eq. (I-41) yields the five positive roots M_j. The shape of the mode pertaining to a particular M_j can be found by computing the ratios $\frac{U}{W}$, $\frac{V}{W}$, $\frac{X}{W}$ and $\frac{Y}{W}$ from any four of the Equations (I-36)- to (I-40). The frequency can be computed from Eq. (I-24). In general, only the lowest frequency ω_1 will be required.

Once the frequency ω and the ratios of the constants have been evaluated for a particular value of n, Eq. (I-25)-Eq. (I-35) may be used to compute the normalized mode shapes, normalized strains and normalized stresses in the shell.

$$\begin{vmatrix}
2M-\frac{6}{\xi^2}-(1-\nu)n^2 & -\frac{3n(3\nu-1)}{2\xi} & 0 & -\frac{3}{2\xi} & \frac{6\nu}{\xi^2} \\
-\frac{3n(3\nu-1)}{2\xi} & 2M-2n^2-\frac{3(1-\nu)}{\xi^2} & \frac{n}{2} & -\frac{n(1+13\nu)}{8} & \frac{3n}{\xi} \\
0 & \frac{n}{2} & \frac{M}{2}-\frac{1}{2} & \nu & 0 \\
-\frac{3}{2\xi} & -\frac{n(1+13\nu)}{8} & \nu & \frac{3}{40}M\xi^2-\frac{19}{8}-\frac{3(1-\nu)n^2\xi^2}{80} & \frac{3\nu}{2\xi} \\
\frac{6\nu}{\xi^2} & \frac{3n}{\xi} & 0 & \frac{3\nu}{2\xi} & \frac{6}{\xi^2}\left\{M-1\right\}
\end{vmatrix} = 0$$

Membrane Effects
5 Constant Approximation
Empty Shell

(I-41)

c) Approximation Using Three Constants - Membrane and Bending Effects. $n \neq 0$

Let the displacements of the shell (Fig. I-1) be given by

$$u(z,\ \theta,\ t) = u(z,\ \theta)e^{i\omega t} = U\frac{z}{a}\cos n\theta\ e^{i\omega t} \tag{I-42}$$

$$v(z,\ \theta,\ t) = v(z,\ \theta)e^{i\omega t} = V\frac{z}{a}\sin n\theta\ e^{i\omega t} \tag{I-43}$$

$$w(z,\ \theta,\ t) = w(z,\ \theta)e^{i\omega t} = W\frac{z}{a}\cos n\theta\ e^{i\omega t} \tag{I-44}$$

Substituting Eq. (I-42)-(I-44) into Eq. (I-7), the kinetic energy of the shell is given by

$$T = \frac{m_i \pi a \omega^2}{6}\ \frac{L^3}{a^2}\left[U^2 + V^2 + W^2\right] \tag{I-45}$$

The potential energy of the shell is again given by Eq. (I-10)-(I-12), which upon substitution of Eq. (I-42)-(I-44) become:

$$V_1 = \frac{Eh\pi}{2(1-\nu^2)}\ \xi \left\{ \begin{aligned} & U^2 + (nV-W)^2\frac{\xi^2}{3} + \nu U(nV-W)\xi \\ & + \frac{(1-\nu)}{2}\left(V^2 + \frac{n^2\xi^2}{3}\ U^2 - n\xi\ UV\right) \end{aligned} \right\} \tag{I-46}$$

and

$$V_2 = \frac{Eh\pi}{24(1-\nu^2)}\ \frac{h^2}{a^2} \left\{ \begin{aligned} & W^2\left[\frac{(1-n^2)^2\xi^3}{3} + 2(1-\nu)n^2\xi\right] \\ & + \left[\left(\frac{1-\nu}{2}\right)\frac{n^2\xi^3}{3}\right]U^2 + \left[\frac{3(1-\nu)}{2}\ \xi\right]V^2 \\ & - \left[\left(\frac{1-\nu}{2}\right)n^2\xi^2\right]UW \left[-\ 3(1-\nu)\xi n\right] VW \end{aligned} \right\} \tag{I-47}$$

Using Eq. (I-15), the following system of three homogeneous equations in the three unknowns U, V and W is obtained:

$$\left[2M - \frac{6}{\xi^2} - n^2(1-\nu)(1+k)\right] U + \left[\frac{-3n(3\nu-1)}{2\xi}\right] V + \left[\frac{3}{\xi}\left\{\nu + \frac{kn^2(1-\nu)}{2}\right\}\right] W = 0 \qquad \text{(I-48)}$$

$$\left[\frac{-3n(3\nu-1)}{2\xi}\right] U + \left[2M - 2n^2 - \frac{3}{\xi^2}(1-\nu)(1+3k)\right] V + \left[2n + \frac{9kn(1-\nu)}{\xi^2}\right] W = 0 \qquad \text{(I-49)}$$

$$\left[\frac{3}{\xi}\left\{\nu + \frac{kn^2(1-\nu)}{2}\right\}\right] U + \left[2n + \frac{9kn(1-\nu)}{\xi^2}\right] V + \left[2M - 2 - \frac{6k}{\xi^2}\left\{(1-n^2)^2 \frac{\xi^2}{3} + 2(1-\nu)n^2\right\}\right] W = 0 \qquad \text{(I-50)}$$

where k and M are defined by Eq. (I-21) and (I-22) respectively. Nonvanishing solutions of the system of Eq. (I-48)-(I-50) and free vibrations exist only if the determinant of the system vanishes. This leads to the determinantal frequency equation shown on Pg. 21.

For given values of ξ and n, Eq. (IV-51) yields three positive roots M_j, defining three mutually orthogonal modes of frequencies

$$\omega_j^2 = \frac{M_j E h}{m_1 a^2(1-\nu^2)} \qquad j = 1, 2, 3. \qquad \text{(I-52)}$$

The shape of the mode pertaining to a particular frequency ω_j can be found by computing the ratios $\frac{U}{W}$ and $\frac{V}{W}$ from any two of Eq. (I-48)-(I-50).

In general, the lowest frequency ω_1 only, will be required and for simplicity in the following expressions, the subscript j will be dropped.

$$\begin{vmatrix} 2M - \frac{6}{\xi^2} - n^2(1-\nu)(1+k) & -\frac{3n(3\nu-1)}{2\xi} & \frac{3}{\xi}\left\{\nu + \frac{kn^2(1-\nu)}{2}\right\} \\ -\frac{3n(3\nu-1)}{2\xi} & 2M - 2n^2 - \frac{3(1-\nu)(1+3k)}{\xi^2} & 2n + \frac{9kn(1-\nu)}{\xi^2} \\ \frac{3}{\xi}\left\{\nu + \frac{kn^2(1-\nu)}{2}\right\} & 2n + \frac{9kn(1-\nu)}{\xi^2} & 2M - 2 - \frac{6k}{\xi^2}\left\{(1-n^2)^2\frac{\xi^2}{3} + 2(1-\nu)n^2\right\} \end{vmatrix} = 0$$

$n \neq 0$

Membrane + Bending Effects
3 Constant Approximation
Empty Tank

(I-51)

The displacements of the shell may be written in the form

$$u(z, \theta) = C_n \frac{U}{W} \frac{z}{a} \cos n\theta \tag{I-53}$$

$$v(z, \theta) = C_n \frac{V}{W} \frac{z}{a} \sin n\theta \tag{I-54}$$

$$w(z, \theta) = C_n \frac{z}{a} \cos n\theta. \tag{I-55}$$

It is convenient to define "normalized modes of vibration" by chosing the constant C_n so that Eq. (I-28) is satisfied. In this case, the modes are normalized to the total mass of the empty shell, $M_1 = 2\pi a L m_1$. Substituting Eq. (I-53)-(I-55) into Eq. (I-28), the normalization coefficient C_n becomes

$$C_n = \left\{ \frac{6}{\ell^2} \left[\frac{1}{1 + (\frac{U}{W})^2 + (\frac{V}{W})^2} \right] \right\}^{1/2} \tag{I-56}$$

Using this value of C_n, Eq. (I-53)-(I-55) give the normalized modes of free vibration of the empty tank.

The results obtained from this approximation may be used where an estimate of the frequency of vibration in any particular mode is required. The work required in solving the third order determinantal frequency equation is materially less than that required for the fifth order determinantal equation. However, the present approximation does not give sufficiently accurate results for the strains and stresses in the shell, and the approximation using five constants must be employed when these quantities are required.

d) Approximation Using Three Constants - Membrane Effects Only.

$n \neq 0$

As in part (b) of this section, a membrane approximation may be used to obtain the frequencies of sufficiently thin shells in the lower modes of n. The frequency equation and the mode shapes for the membrane shell are obtained by setting the value of the constant k equal to zero in Eq. (I-48)-(I-51). The determinantal frequency equation is given on Page 24.

The remarks of part (c) of this section regarding the validity and applicability of three constant approximation also hold for this case.

$$\begin{vmatrix} 2M - \frac{6}{\xi^2} - n^2(1-\nu) & -\frac{3n(3\nu-1)}{2\xi} & \frac{3\nu}{\xi} \\ -\frac{3n(3\nu-1)}{2\xi} & 2M - 2n^2 - \frac{3(1-\nu)}{\xi^2} & 2n \\ \frac{3\nu}{\xi} & 2n & 2M-2 \end{vmatrix} = 0$$

$n \neq 0$

Membrane Effects
3 Constant Approximation
Empty Tank

(I-57)

e) Mode n = 0

It is shown in Reference (1)[7] that the exact solution of the boundary value problem for the free vibrations of a thin cylindrical shell in the mode n = 0 yields an infinite number of frequency numbers M_{oj} lying between closely spaced limits. For a steel shell ($\nu = 0.3$) with $\xi = 0.8$, these limits are given by $M = 0.8833$ and 0.91. Since the frequencies and their corresponding sub-modes are so close together, a forced vibration analysis for the response of the shell in the mode n = 0 would require an expansion involving, in general, many of these sub-modes of frequency ω_{oj}. This is not practicable and in general, other methods must be employed to approximate the response of the shell in the zero mode. This problem is discussed in some detail in Section VI of this Report. One suitable method is to consider the shell as a series of separate rings.

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(7) Reference (1), DiMaggio, Pg 12 and 13

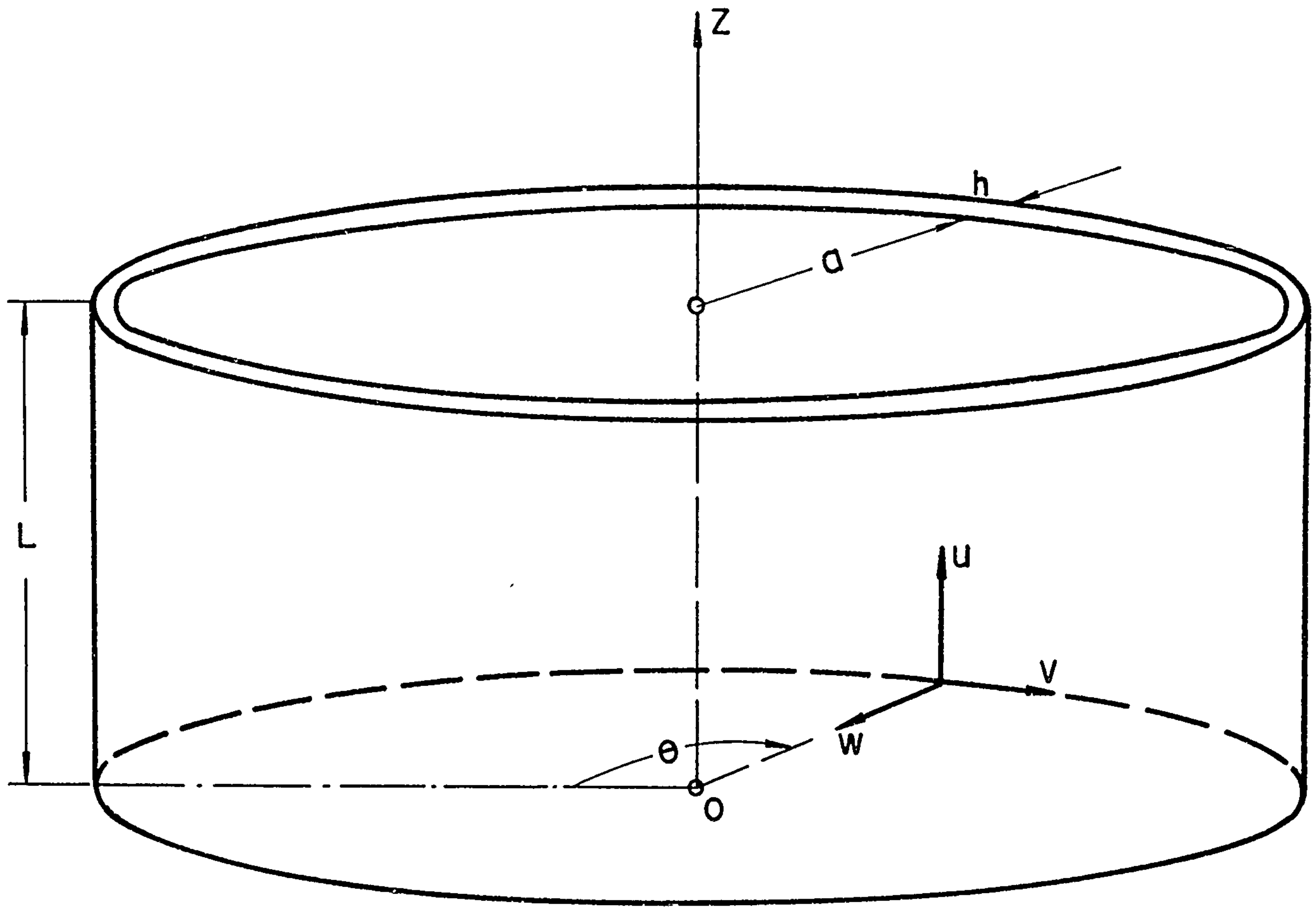

FIG. I-1 GEOMETRY OF THE PROBLEM

II Comparison of Results for Membrane Shells

The values of the frequency number M_n, and the normalized strains $a\epsilon_{\theta\theta}$, $a\epsilon_{zz}$, $a\epsilon_{z\theta}$ obtained from the five constant membrane approximation of Section I, Part b, will be compared with the results of the membrane analysis of Reference (1). An empty steel shell in which $\xi = 0.8$, $\nu = 0.3$ and $\frac{h}{a} = \frac{1}{1200}$ will be considered. For the modes $1 \leq n \leq 9$, the shell acts as a membrane, as shown in Fig. (III-1).

Table (II-1) gives the computed values of the lowest frequency number, M_{n1}, for the modes $1 \leq n \leq 6$ of the steel tank under consideration. The results of both the five constant and the three constant approximations are given and these results are compared with the values of M_{n1} from the membrane theory of Reference (1). In addition, the percentage error in the frequency, ω_{n1}, is given for the results of both approximations.

It is seen that the error in the frequency ω_{n1} varies between 1.6% and 6.3% for the five constant approximation, and between 4.3% and 19.1% for the three constant approximation. It is concluded that both approximate theories give sufficiently accurate estimates of the frequencies of vibration of the shell for use in problems concerning the response of the shell to dynamic loading. It should be noted that the increased accuracy of the five constant approximation is obtained at the cost of the extra labor involved in expanding a fifth order determinant rather than one of the third order.

For applications in which the values of the strains and stresses of the shell in each mode n are required, the five constant theory must be used. It should be noted that although the frequencies of vibration

and the shape of the modes, $u(\theta, z)$, $v(\theta, z)$ and $w(\theta, z)$ can be estimated by means of the three constant theory, the strains and corresponding stresses which involve derivatives of the displacements cannot be obtained from this theory with sufficient accuracy for use in forced vibration problems.

On the other hand, the five constant theory gives sufficiently accurate values of the shell strains and stresses in the range of particular interest. For future applications, the longitudinal stress σ_{zz} at the bottom of the shell will be of particular interest in analyzing the uplifting of the shell under dynamic loading. Also of interest will be the direct stress $\sigma_{\theta\theta}$ at the top of the shell for possible use in connection with both buckling considerations. The determination of these stresses requires the values of the strains $\epsilon_{\theta\theta}$ and ϵ_{zz} at the top and bottom of the tank.

Fig. (II-1)-(II-4) show the variation of the strain $a\epsilon_{zz}$ with the height of the tank for the modes n = 1, 2, 4 and 6. The strains obtained from both the five constant theory and the membrane analysis of Reference (1) are shown. It is felt that the results of the five constant theory are of sufficient accuracy for use in applications to forced vibration problems.

Fig. (II-5)-(II-8) show the variation of the strain $a\epsilon_{\theta\theta}$ with the height of the tank for the modes n = 1, 2, 4, 6. Again, it is felt that the results of the five constant theory are of sufficient accuracy for use in application to forced vibration problems.

Fig. (II-9)-(II-12) give the values of the shear strain $a\epsilon_{z\theta}$ for the modes n = 1, 2. 4 and 6. It is seen that the five constant approximation

gives a poor result for this strain. However, as this strain is not of any practical importance in our applications, this is no deterrent to the use of the five constant theory. A better value of a $\epsilon_{z\theta}$ could be obtained from a six or possibly seven constant theory, but this would lead to higher order frequency determinants and would serve no useful purpose in future applications.

Table II-1

Frequency Number, M_n

Empty Steel Tank $L/a = 0.8$

$\nu = 0.3$

$\frac{h}{a} = \frac{1}{1200}$

n	Five Constant Approximation		Three Constant Approximation		Reference (1)
	M_{n1}	Error in ω_{n1} (%)	M_{n1}	Error in ω_{n1} (%)	M_{n1}
1	.4575	3.9	.4610	4.3	.4242
2	.1906	2.9	.1974	4.7	.1802
3	.0859	1.5	.0934	5.8	.0834
4	.0432	1.6	.0494	8.8	.0418
5	.0241	3.4	.0290	13.3	.0226
6	.0147	6.3	.0184	19.1	.0130

$$\omega_{n1}^2 = \frac{M_{n1} Eh}{m_1 a^2 (1-\nu^2)}$$

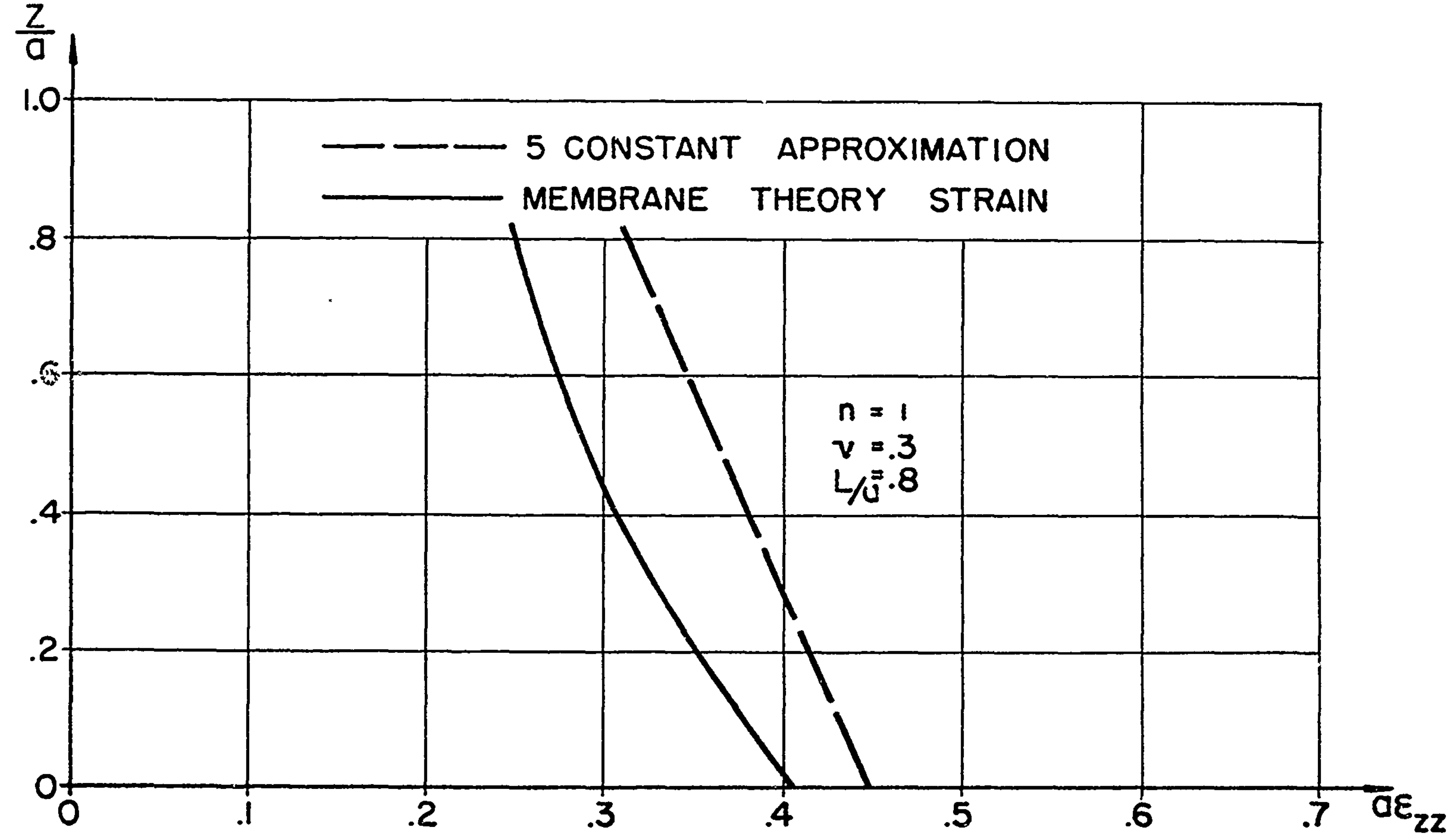

FIG. II-1 NORMALIZED MEMBRANE STRAINS–$a\varepsilon_{zz}$

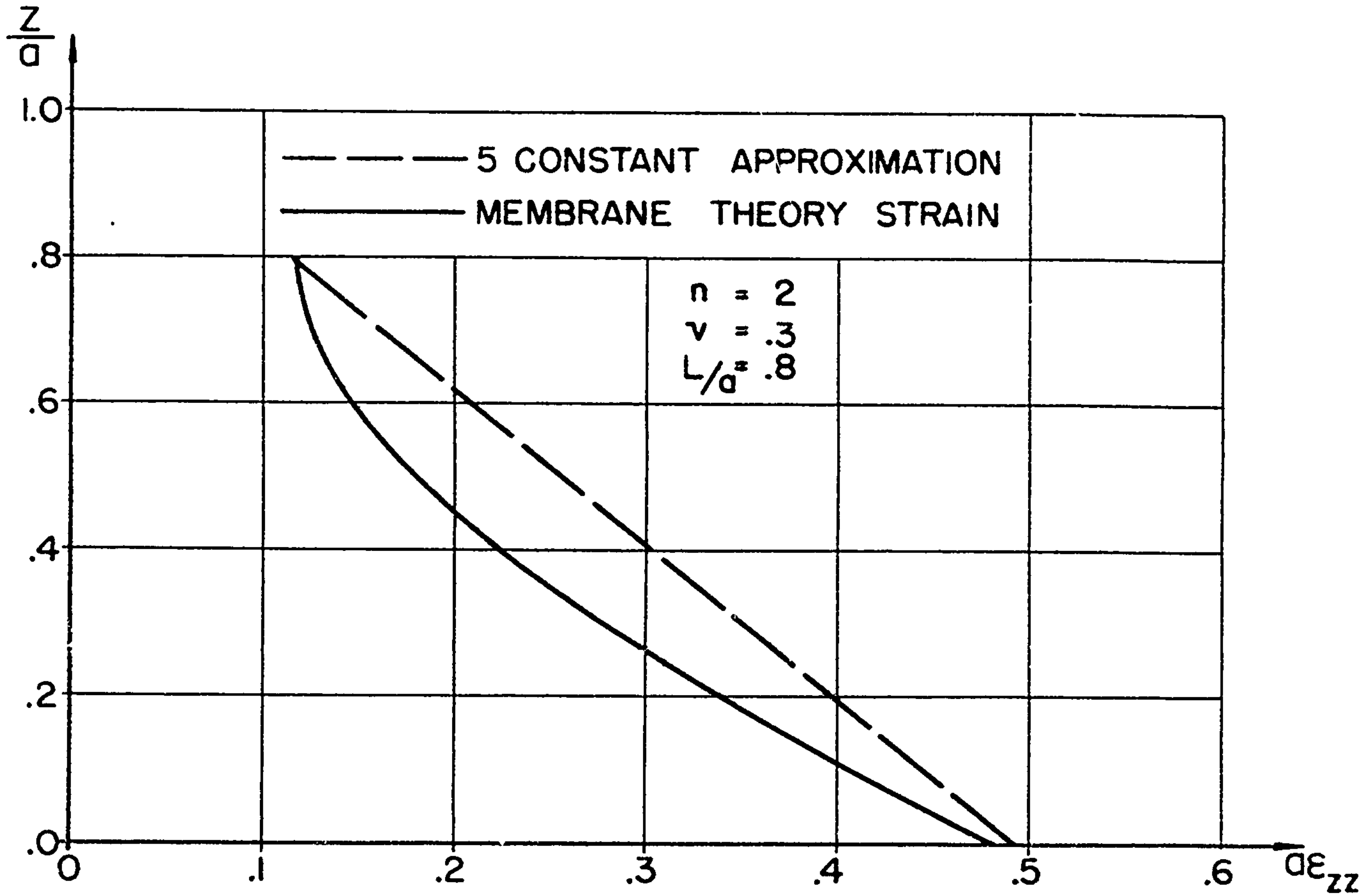

FIG.II-2 NORMALIZED MEMBRANE STRAINS

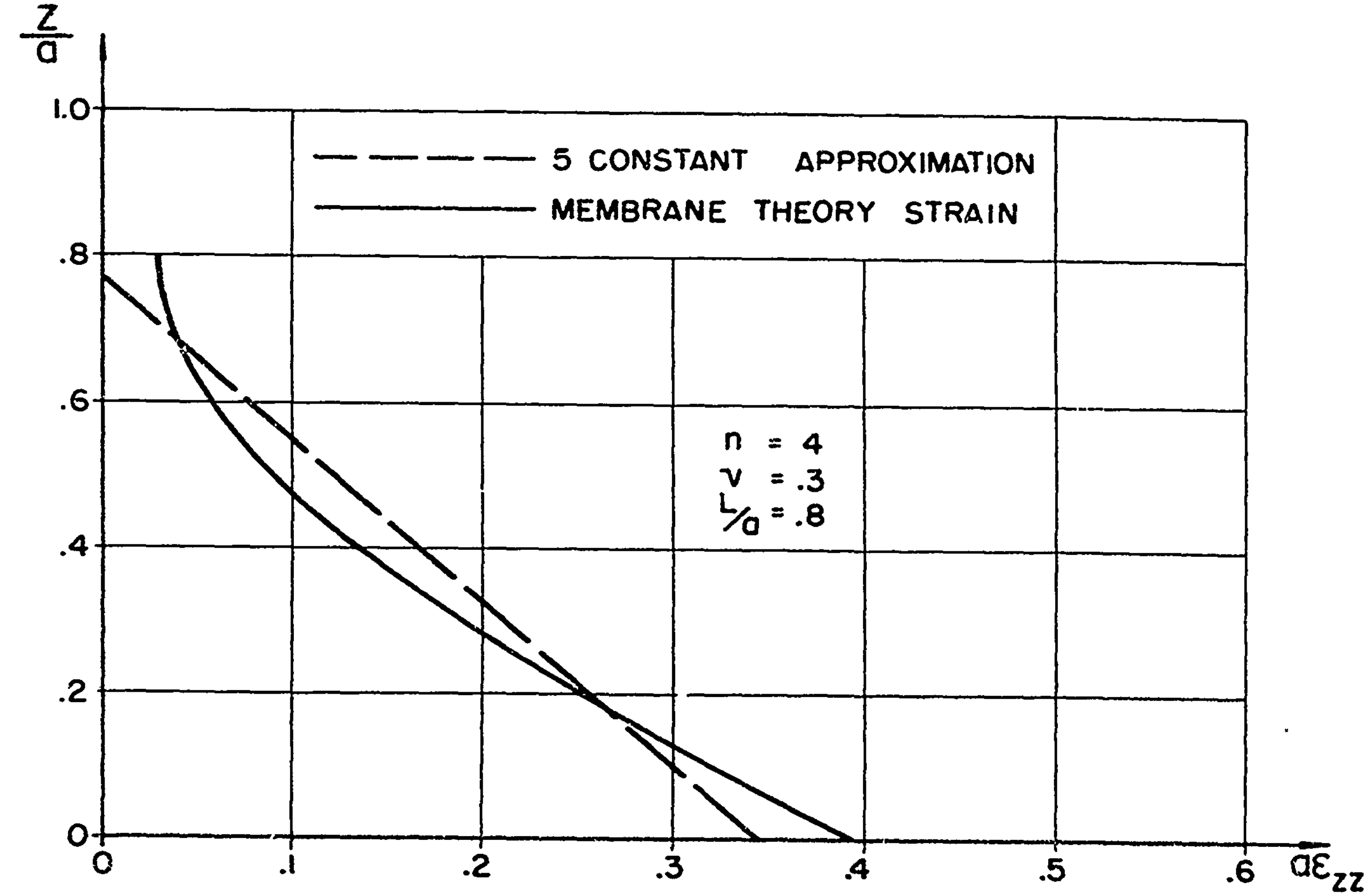

FIG. II-3 NORMALIZED MEMBRANE STRAINS

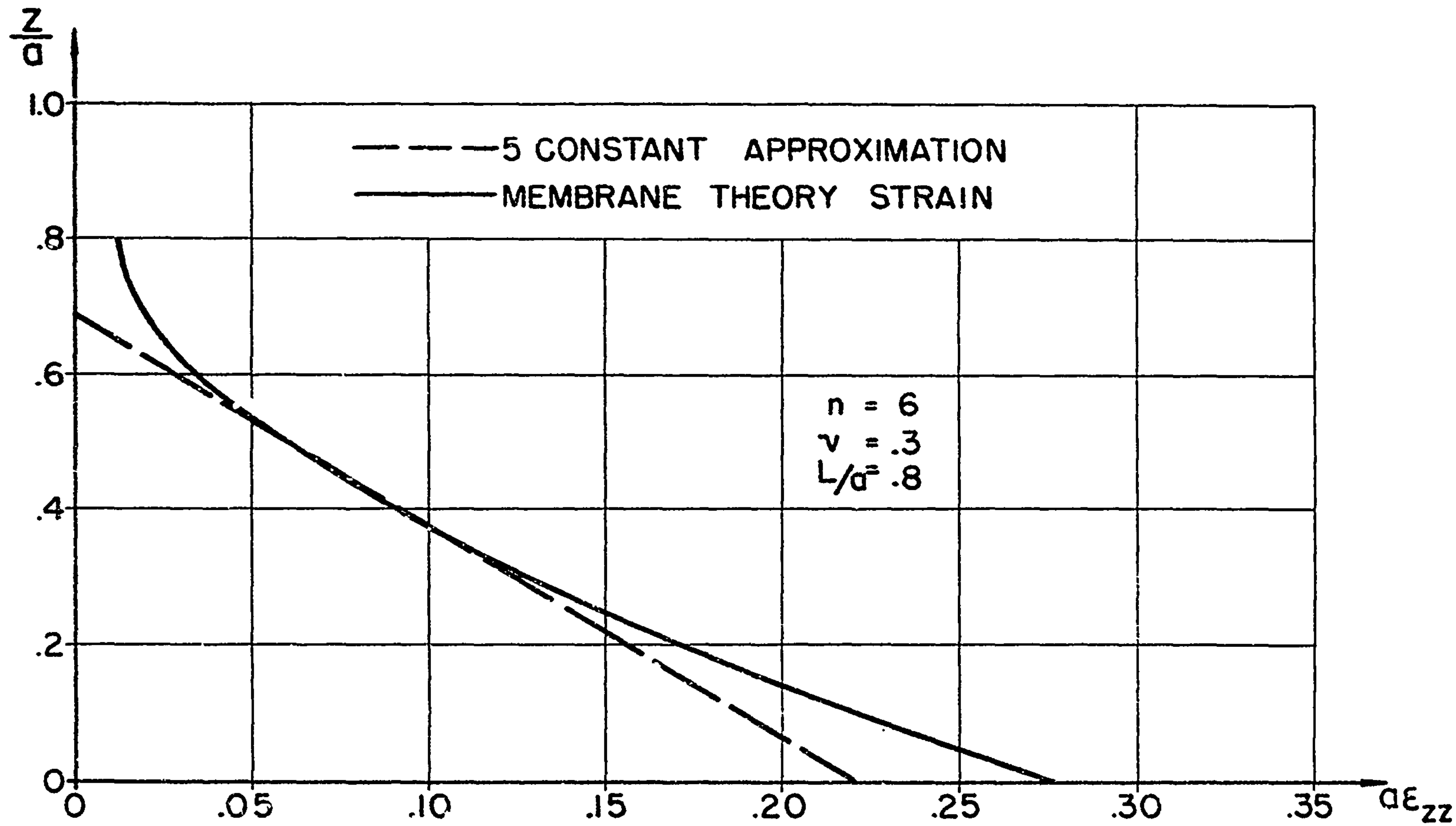

FIG. II-4 NORMALIZED MEMBRANE STRAINS-$a\varepsilon_{zz}$

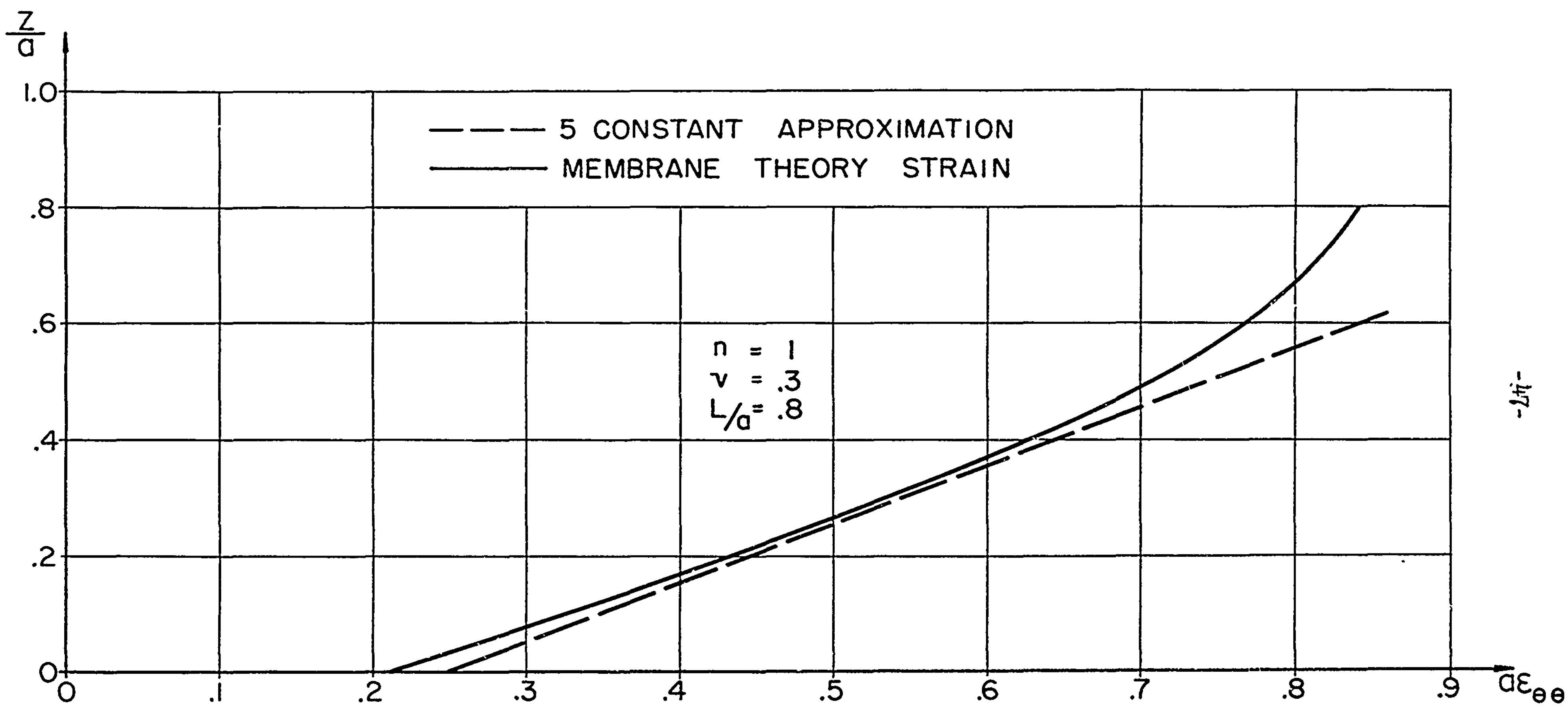

FIG. II-5 NORMALIZED MEMBRANE STRAINS

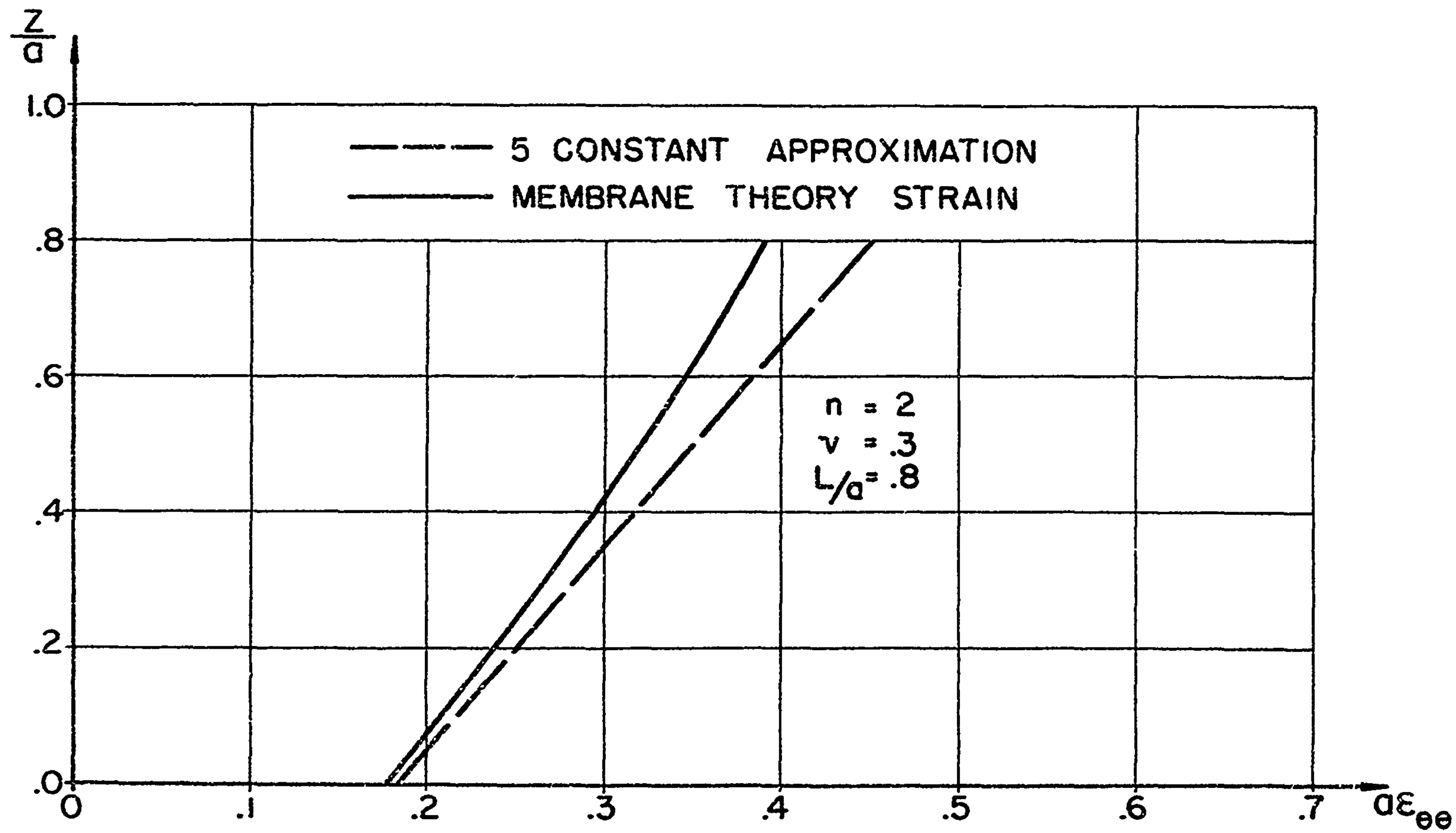

FIG. II-6 NORMALIZED MEMBRANE STRAINS

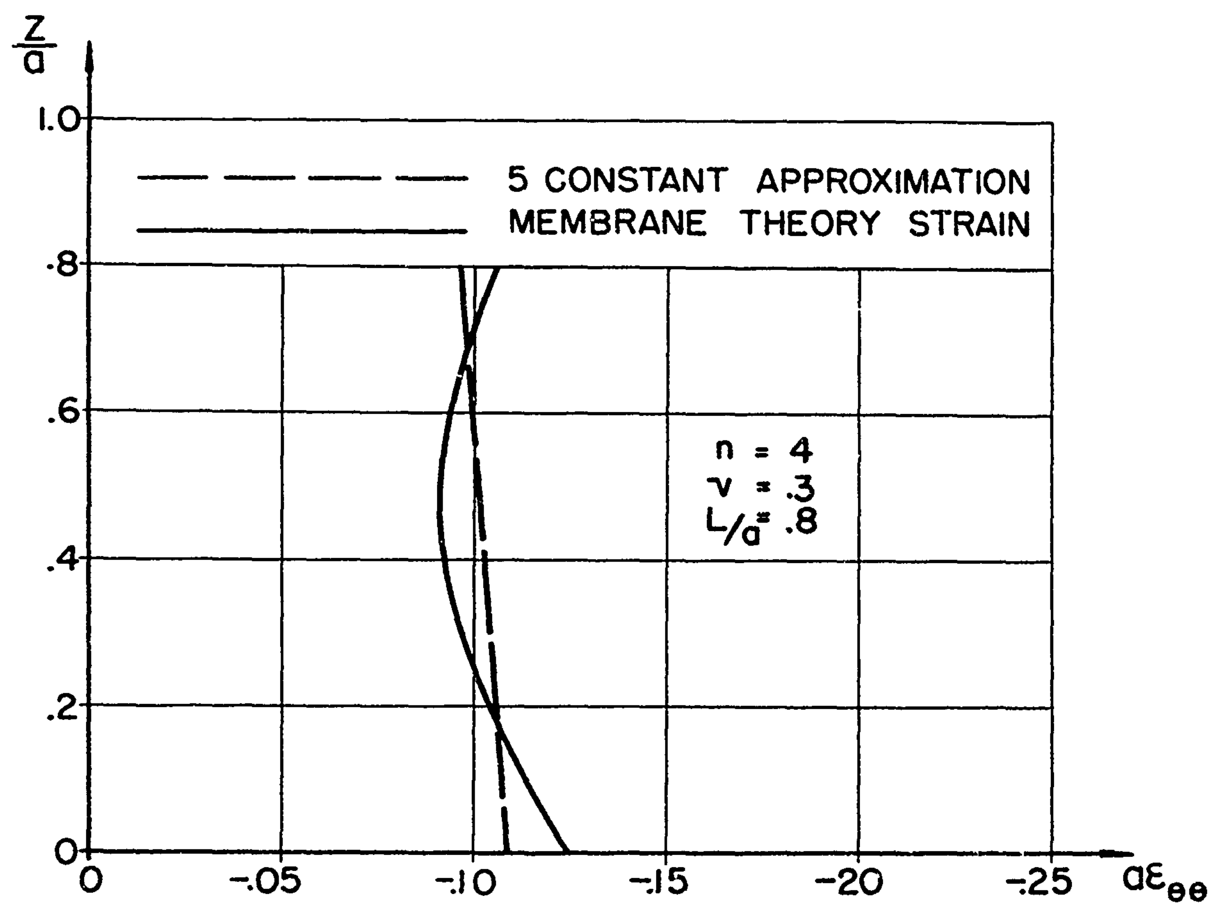

FIG.II-7 NORMALIZED MEMBRANE STRAINS

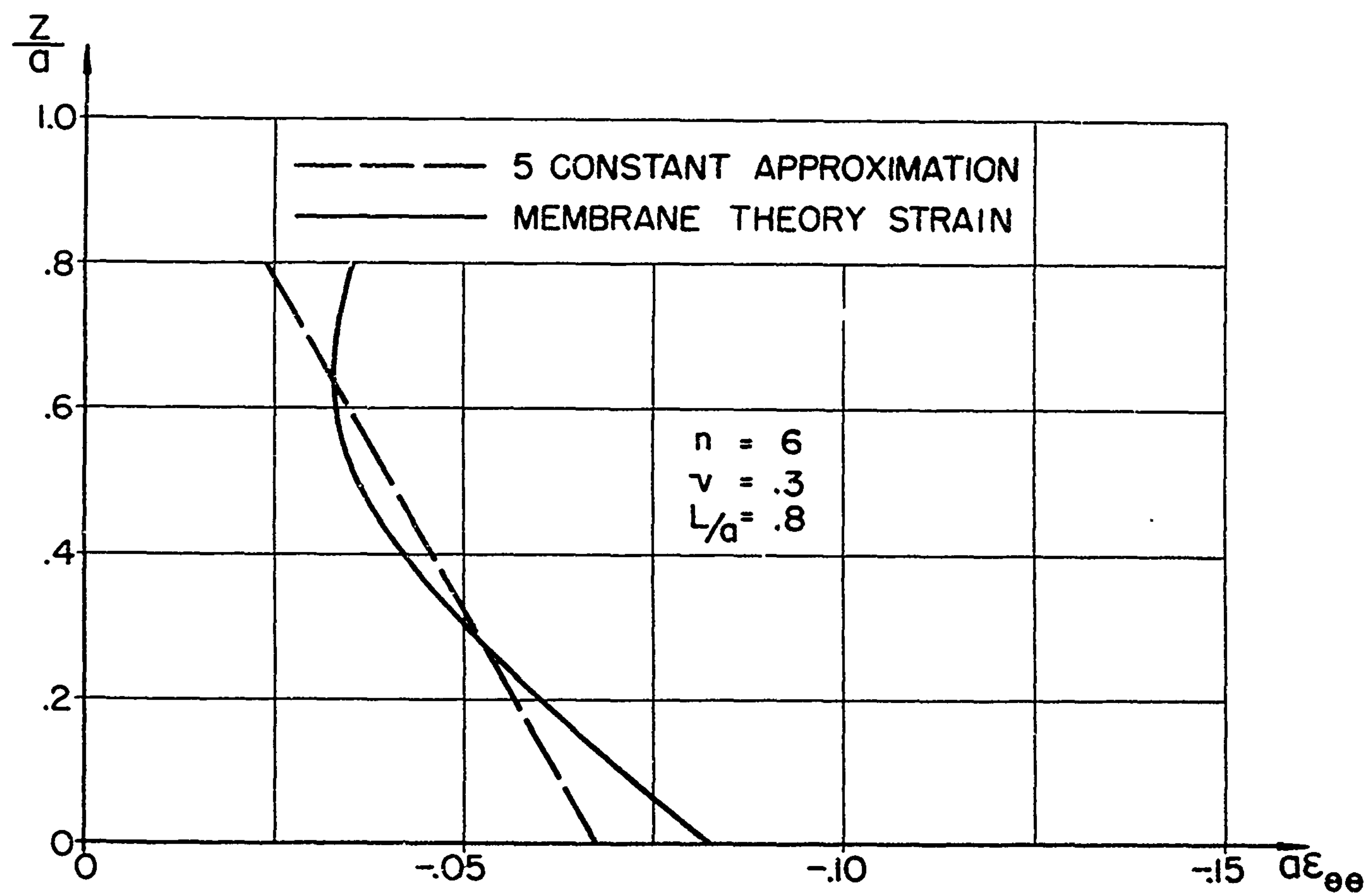

FIG.II-8 NORMALIZED MEMBRANE STRAINS

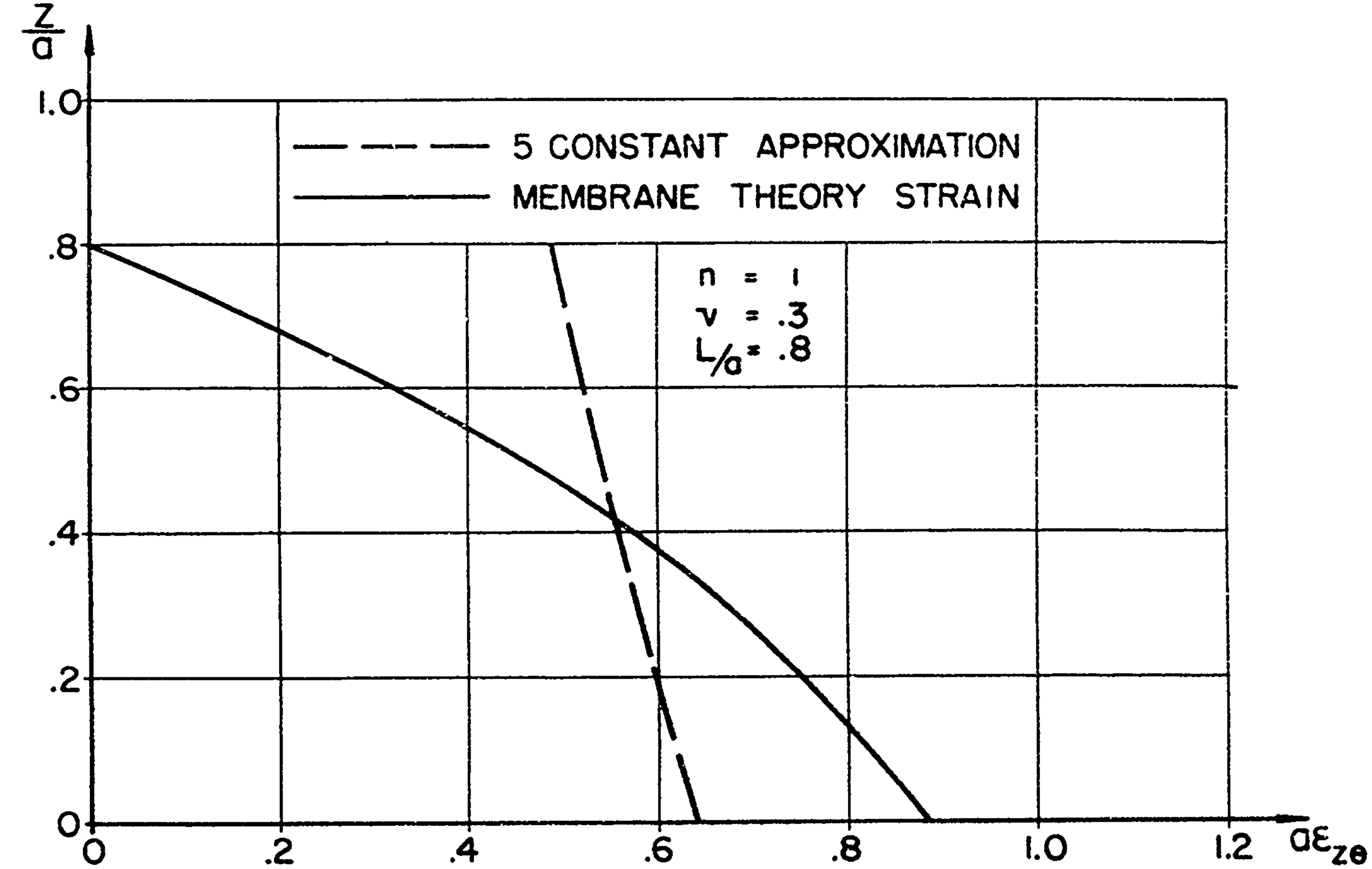

FIG. II-9 NORMALIZED MEMBRANE STRAINS

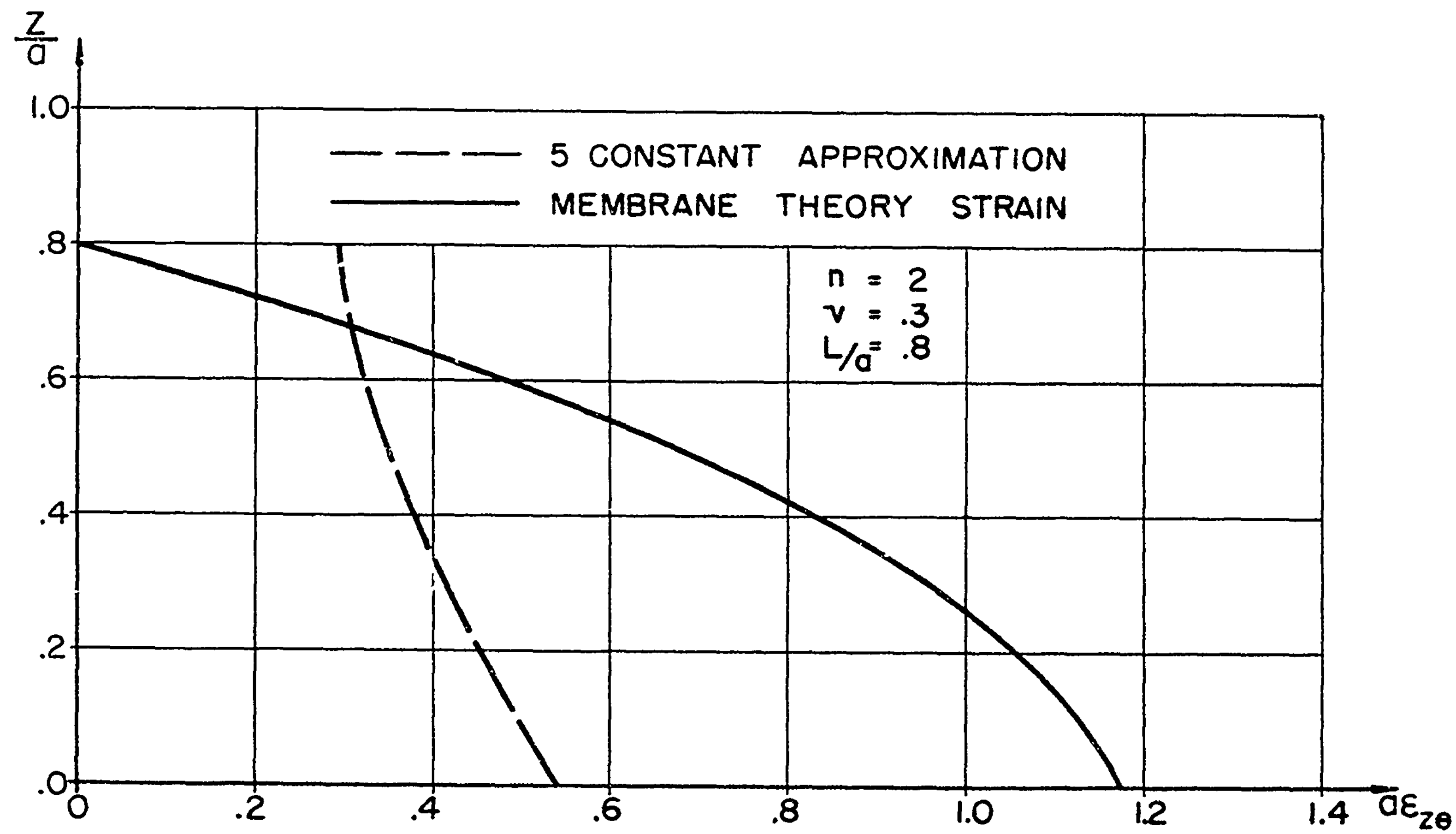

FIG.II-10 NORMALIZED MEMBRANE STRAINS

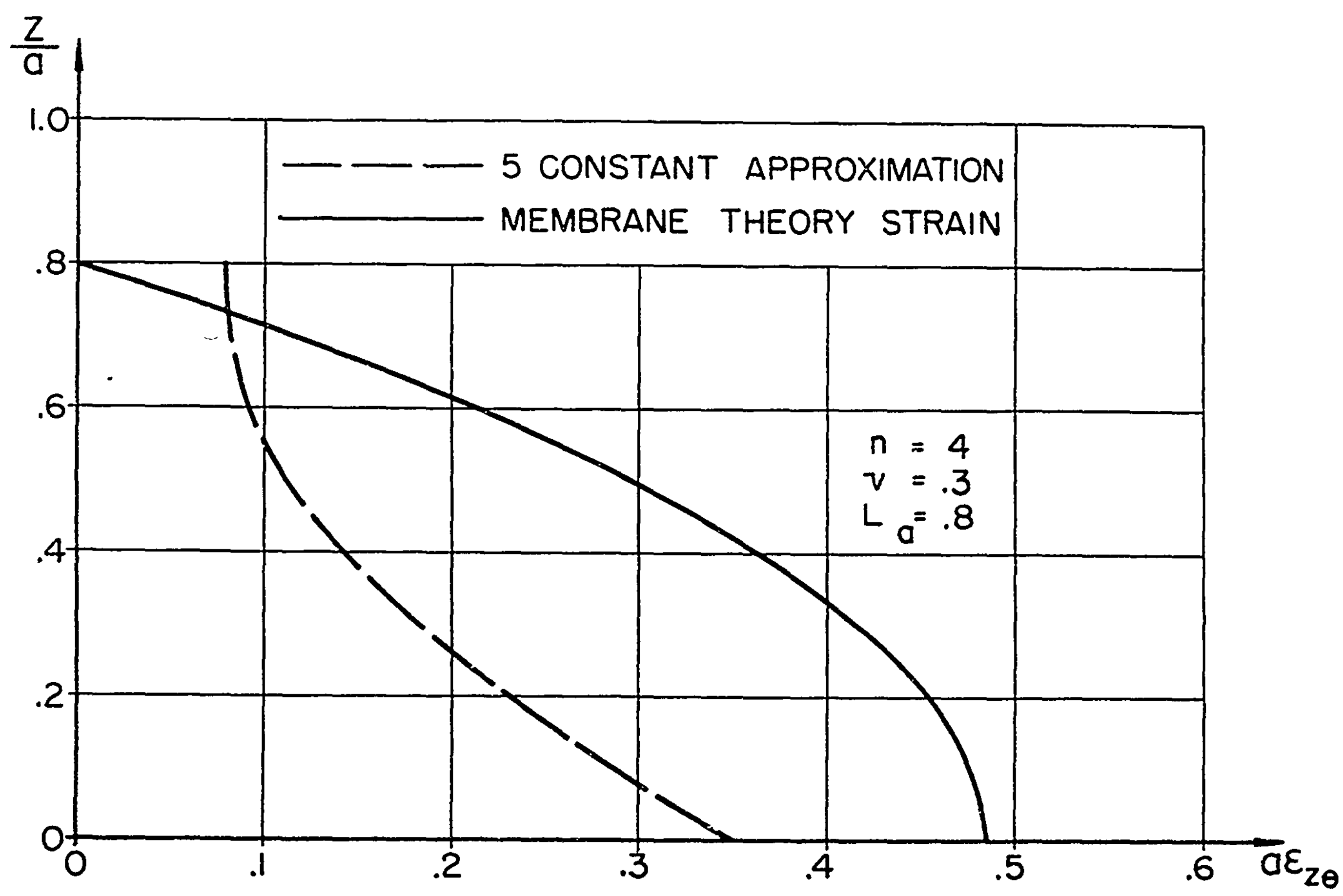

FIG.II-11 NORMALIZE MEMBRANE STRAINS

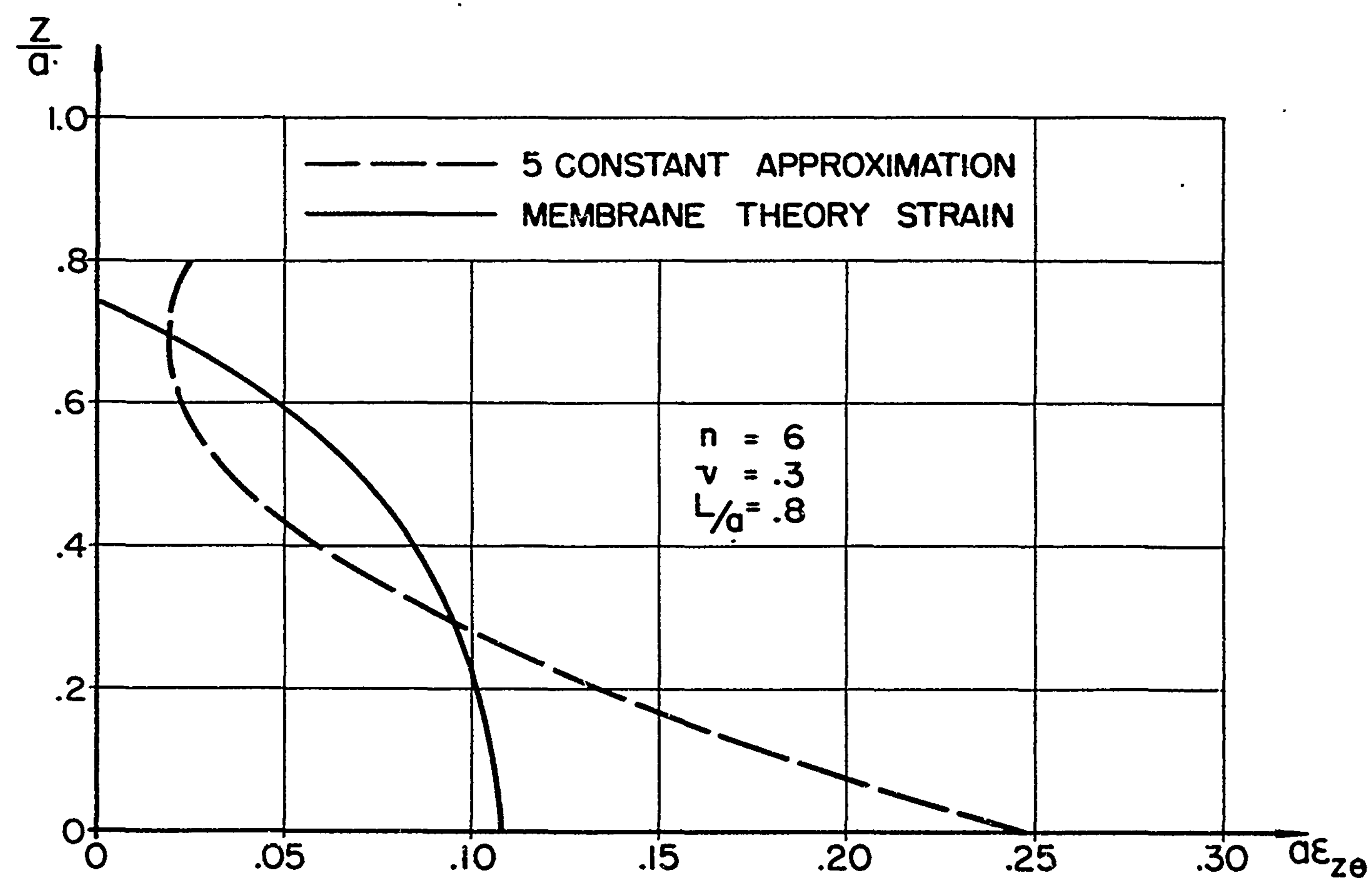

FIG.II-12 NORMALIZED MEMBRANE STRAINS

III Bending Effects on the Frequencies and Modes of Free Vibrations of Thin Cylindrical Shells.

This section presents a study of the effect of bending strains on the frequencies and modes of free vibrations of thin cylindrical shells. It should be noted that for shells which are sufficiently thin to be considered membrane shells, the frequencies of free vibrations are independent of the thickness to radius ratio, h/a, and depend only on the height to radius ratio, $\xi = L/a$. For a shell of a given material and ratio ξ lying in this range, the membrane frequency would be the same regardless of the thickness of the shell. The problem therefore takes on two aspects for discussion:

1. The establishment of ranges in which shells of practical interest could be considered as membranes.

2. The derivation of a simple procedure which would give an estimate of the effect of bending strains on the lowest membrane frequency in any mode n. Such a procedure would provide a simple method for establishing the range of modes in which a given shell could be considered to be a membrane. It would also provide an estimate of the effect of bending strains on the lowest membrane frequency in any mode n, in the range in which bending effects are of importance. The advantage of obtaining such information without the necessity of expanding the determinantial frequency equation, Eq. (I-23), for each shell with a different h/a ratio is obvious.

It should be noted however, that if values of the strains and stresses in the shell are also required, the determinantal equation, Eq. (I-23), must be expanded to obtain the lowest root M_{n1}, and the corresponding mode

shapes must be computed from Eq. (I-16)-(I-20). The use of the mode shapes resulting from membrane theory does not give an accurate determination of the strains in the shell in the range in which bending effects are of importance.

An approximation for the determination of the frequencies and mode shapes of thin cylindrical shells with small $\frac{h}{a}$ ratios was given in Section I, parts (b) and (d), by considering the shell to be a membrane with no bending stiffness. The results of such an approximation were noted to be valid for modes with a low circumferential wave number n.

In order to establish a range of validity of $\frac{h}{a}$ and n for this approximation [(1)] and to study the effects of bending strains on the frequencies of the shell, computations were made and are presented for the following two shells of practical interest:

a) Unprotected steel tank, $\frac{h}{a} = \frac{1}{1200}$, $\frac{L}{a} = 0.8$, $\nu = 0.3$

b) Steel tank with concrete shielding, $\frac{h}{a} = \frac{1}{31}$, $\frac{L}{a} = 0.8$, $\nu = 0.3$

Figure (III-1) shows a curve of the frequency number M_n (corresponding to the lowest frequency j = 1 in each mode n) plotted against the mode number n for the unprotected steel tank. The values of M_{n1} on the curve labeled "membrane only" were derived from the membrane frequency equation, Eq. (I-41), while those on the curve labeled "membrane plus bending" were derived from the complete frequency equation, Eq. (I-23). For a shell of this relatively small thickness to radius ratio, both theories give the

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(1) This range is quite different for steel tanks and for steel tanks with concrete shielding.

same frequencies in the range $1 \leq n \leq 9$. For the modes in which $9 \leq n \leq 14$, the inextensional bending strains and the extensional membrane strains have about equal effects on the frequency. For $n > 14$, the bending strains become predominant and control the frequencies of vibration of the shell.

It should be noted therefore, that if these modes are to be used in the analysis of the forced vibrations of the empty steel tank to dynamic loading, the membrane theory can be used and bending effects may be neglected. In such an analysis, practical considerations of the convergence of the mode series for the response of the shell to dynamic loadings, would certainly not require more than the number of accurate modes given by the membrane analysis.

The situation changes radically when the steel tank is protected by a concrete shielding. Fig. (III-2) shows the curve of M_n versus n for this case. As in Fig. (III-1), the "membrane only" and "membrane plus bending" curves are computed from Eq. (I-41) and Eq. (I-23) respectively. Since the thickness to radius ratio of the protected shell is roughly forty times larger than that for the unprotected shell, it is to be expected that the bending effects will become of importance at a much lower mode n than in the case of the unprotected shell. It is seen from Fig. (III-2) that a membrane analysis gives accurate results in the range $1 \leq n \leq 3$. Bending effects start to become of importance at $n = 3$ and for $n > 4$, the bending strains become predominant and control the frequency of vibration of the shell. The forced vibration analysis of the response for a protected steel tank to dynamic loading would therefore

require that bending effects be included in all modes where $n > 3$.

A simple estimate of the effect of bending strains on the membrane frequencies of a cylindrical shell will now be developed. This correction to the membrane frequency will enable the computation of the frequencies of vibration of shells in the range where bending energy is of importance, without resorting to an expansion of the fifth order determinantal frequency equation, Eq. (I-23), for each shell with a different h/a ratio. Moreover, for a shell with a particular thickness to radius ratio h/a, it will enable us to determine the range in which the shell may be considered to be a membrane.

To obtain this bending correction to the membrane frequencies of a shell, consider the case of purely inextensional motions of a thin cylindrical shell in free vibrations. The condition of inextension of the middle surface of the thin shell requires that the strain $a\epsilon_{\theta\theta} = V_\theta - W$ be equal to zero. This condition is satisfied by displacements of the form

$$w(z, \theta, t) = \frac{z}{a} \cos n\theta \; e^{i\omega_I t} \tag{III-1}$$

$$v(z, \theta, t) = \frac{1}{r_c} \frac{z}{a} \sin n\theta \; e^{i\omega_I t} \tag{III-2}$$

$$u(z, \theta, t) = -\frac{1}{n^2} \cos n\theta \; e^{i\omega_I t} \tag{III-3}$$

The frequency of these inextensional oscillations can be determined by Rayleigh's Principle:

$$T_{max} = V_{max} \tag{III-4}$$

Substituting Eq. (III-1)-(III-3) into the strain energy expression for V_2^* as given by Eq. (I-12) the potential energy of the shell is given by

$$V_{2max} = \frac{Eh\pi}{24(1-\nu^2)} \quad \frac{h^2}{a^2}\,\ell \left[\left\{2(1-\nu) + \frac{n^2\ell^2}{3}\right\}\left(\frac{1-n^2}{n}\right)^2\right]. \qquad \text{(III-5)}$$

while the maximum kinetic energy is obtained by substitution of these equations into Eq. (I-7):

$$T_{max} = \frac{m_1\pi a^2\omega_I^2}{2}\,\ell \left[\frac{\ell^2}{3}\left(\frac{n^2+1}{n^2}\right) + \frac{1}{n^4}\right] \qquad \text{(III-6)}$$

Substituting Eq. (III-5) and (III-6) into Eq. (III-4) the frequency number M_{nI} is given by the relation

$$M_{nI} = \frac{h^2}{12a^2} \quad \frac{n^2(1-n^2)^2}{n^2+1} \left[\frac{2(1-\nu) + \dfrac{n^2\ell^2}{3}}{\dfrac{n^2\ell^2}{3} + \dfrac{1}{n^2+1}}\right] \qquad \text{(III-7)}$$

where M_{nI} is defined by Eq. (I-24). The corresponding frequency ω_{nI} becomes:

$$\omega_{nI}^2 = \frac{Eh^2}{12\rho_1(1-\nu^2)a^4} \quad \frac{n^2(n^2-1)^2}{n^2+1} \left[\frac{1+\dfrac{6(1-\nu)}{n^2\ell^2}}{1+\dfrac{3}{(n^2+1)n^2\ell^2}}\right] \qquad \text{(III-8)}$$

Eq. (III-7) and (III-8) give the frequencies of the purely inextensional motions of thin cylindrical shells.

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(2) The potential energy of extension, V_1 is identically equal to zero for these motions.

To obtain an estimate of the effect of bending strains on the lowest frequency factor M_{n1} in any mode n_1 the value of M_{n1} for the membrane shell only is computed from Eq. (I-41) or Eq. (I-57). Let the quantity be called M_{nm}. The value of the frequency factor M_{nI} for purely inextensional motions of the shell is computed from Eq. (III-7). Noting that the factors M are proportional to the square of the frequency, an approximate value of the frequency factor M_n which includes both membrane and bending effects is obtained from the relation:

$$M_n = M_{nm} + M_{nI} \tag{III-9}$$

The corrected frequency w_n is then given by the relation

$$\omega_n = \left[\omega_{nm}^2 + \omega_{nI}^2 \right]^{1/2} \tag{III-10}$$

where ω_{nm} and ω_{nI} are computed from Eq. (I-24) and Eq. (III-8) respectively. The method used in obtaining Eq. (III-10) is sometimes known as Southwell's method.

Tables (III-1) and (III-2) show the application of Eq. (III-9) to the cases of the protected and unprotected steel tanks respectively. Column 4 of each table gives the value of M_n computed from Eq. (III-9) while column 5 gives the value obtained by expanding the fifth order determinantal frequency equation, Eq. (I-23). It may be noted that very good approximations to the frequency factor M_n can be obtained using Eq. (III-9). In addition, the range of applicability of the membrane theory can easily be established from such tables, thus verifying the results shown in Fig. (III-1) and Fig. (III-2).

It should again be noted however, that when strains and stresses are required in a mode in which bending effects are important, the value of M_n should be obtained by expanding Eq. (I-23) and the correct mode shapes should be evaluated using Eq. (I-16)-(I-20).

Table III-1

Effect of Bending on Membrane Frequencies

Empty Steel Tank - Concrete Shielding
Five Constant Approximation
$L/a = 0.8, \frac{h}{a} = \frac{1}{31}$

n	M_{nm}	M_{nI}	M_n Eq. (III-9)	M_n Eq. (I-23)
1	.4575	0	.4575	.4575
2	.1906	.0004	.1910	.1911
3	.0859	.0075	.0934	.0934
4	.0432	.0233	.0665	.0659
5	.0241	.0550	.0791	.0779
6	.0147	.1112	.1259	.1240

Table III-2

Effect of Bending on Membrane Frequencies

Empty Steel Tank - No Shielding
Five Constant Approximation
$L/a = 0.8, \frac{h}{a} = \frac{1}{1200}, \nu = 0.3$

n	M_{nm}	M_{nI}	M_n Eq. (III-9)	M_n Eq. (I-23)
5	.02412	.00004	.02416	.02412
6	.0147	.0001	.0148	.0147
10	.00360	.00060	.00420	.00420
12	.00222	.00123	.00345	.00350
15	.00124	.00298	.00422	.00421
20	.00061	.00934	.00995	.00990

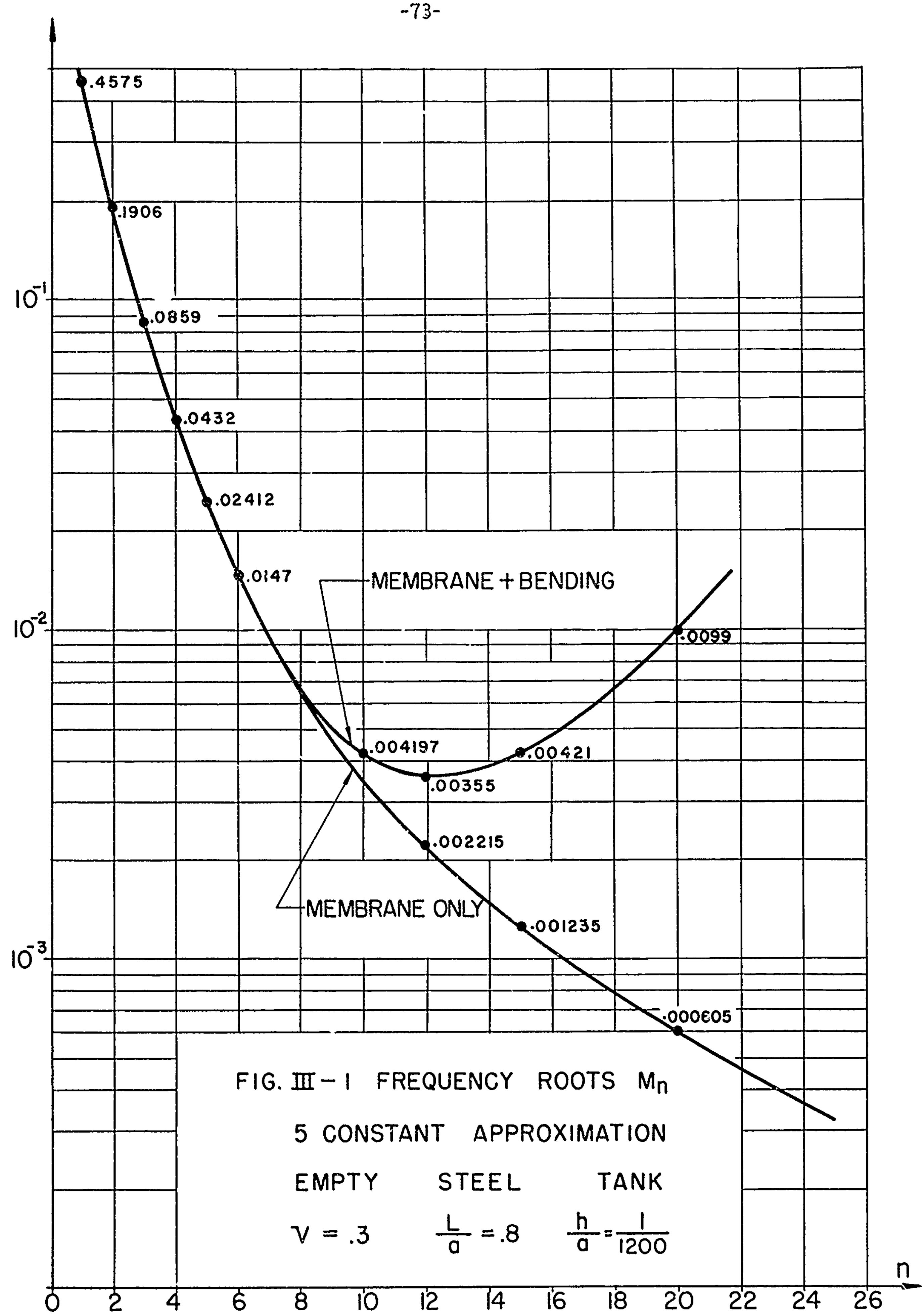

FIG. III - 1 FREQUENCY ROOTS M_n

5 CONSTANT APPROXIMATION

EMPTY STEEL TANK

$\nu = .3 \quad \frac{L}{a} = .8 \quad \frac{h}{a} = \frac{1}{1200}$

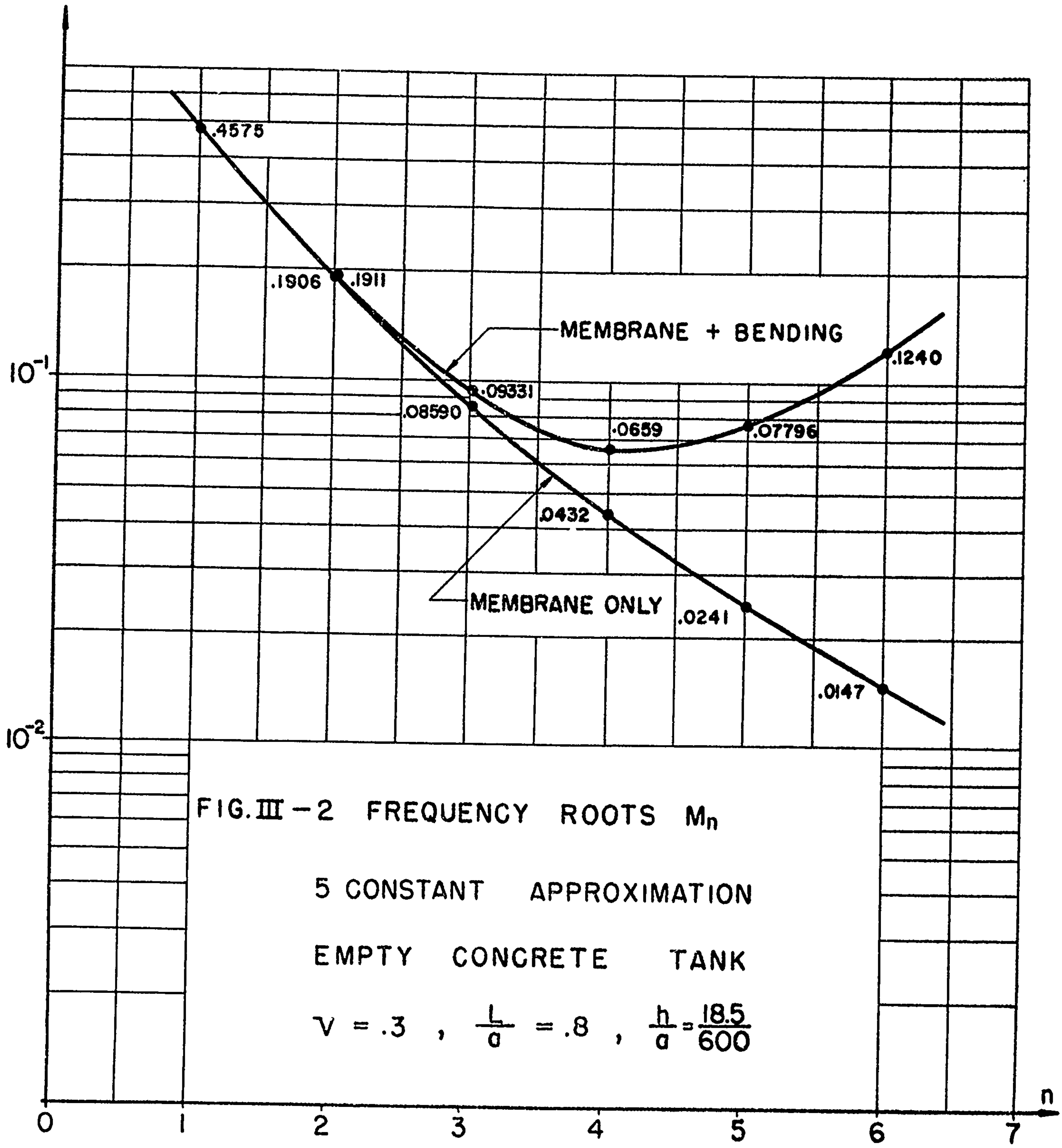

FIG. III – 2 FREQUENCY ROOTS M_n

5 CONSTANT APPROXIMATION

EMPTY CONCRETE TANK

$\nu = .3 \quad , \quad \frac{L}{a} = .8 \quad , \quad \frac{h}{a} = \frac{18.5}{600}$

IV Free Vibrations of Fluid Filled Shells- Determination of the Virtual Mass of the Fluid

Methods leading to the determination of the frequencies and modes of free vibration of empty cylindrical fuel tanks have been presented in previous sections of this report. In this section, the analysis will be extended to the case of partially full cylindrical fuel tanks. As in the case of the empty tank, the modes of vibration will be characterized by the parameter n, the number of circumferential waves in the mode.

Consider the shell of Figure(IV-1) which is filled to a height γL with an incompressible inviscid fluid of density ρ.. The assumption of the incompressibility and zero viscosity of the fluid implies the existence of a velocity potential function, ϕ (r,θ, z, t), such that

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{(IV-1)}$$

The motion of the fluid may be derived from ϕ, since the velocity of the fluid in a direction s is given by the directional derivative

$$V_s = -\frac{\partial\phi}{\partial s} \qquad \text{(IV-2)}$$

The expressions for the radial, tangential and longitudinal velocities of the fluid particles in polar coordinates are

$$V_r = -\frac{\partial\phi}{\partial r} \qquad \text{(IV-3)}$$

$$V_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} \qquad \text{(IV-4)}$$

$$V_z = -\frac{\partial\phi}{\partial z} \qquad \text{(IV-5)}$$

The pressure in the fluid at any point in space is given by the derivative of ϕ with respect to time,

$$p = \rho \frac{\partial \phi}{\partial t} \qquad \text{(IV-6)}$$

where it has been assumed that the velocities are small.

The motion of the fluid during the free vibrations of the partially filled shell can be obtained by the solution of the boundary value problem involving Eq. (IV-1) and the following three boundary conditions:

a) The radial velocity v_r of the fluid must be equal to the radial velocity $\dot{w}$ of the shell on the surface $r = a$.

$$\dot{w}(z, \theta, t) = - \left. \frac{\partial \phi(r, \theta, z, t)}{\partial r} \right]_{r = a} \qquad \text{(IV-7)}$$

b) The longitudinal velocity v_z at the bottom of the shell, $z = - \frac{\gamma L}{2}$, must be equal to zero.

$$v_z = - \left. \frac{\partial \phi(r, \theta, z, t)}{\partial z} \right]_{z = - \frac{\gamma L}{2}} = 0 \qquad \text{(IV-8)}$$

and

c) On the free surface at $z = + \frac{\gamma L}{2}$, the pressure must be equal to zero:

$$p = \rho \left. \frac{\partial \phi(r, \theta, z, t)}{\partial t} \right]_{z = \frac{\gamma L}{2}} = 0 \; . \qquad \text{(IV-9)}$$

Eq. (IV-1), (IV-7), (IV-8), (IV-9) enable the determination of the velocity potential ϕ and subsequently of the kinetic energy of the fluid from the relation:

$$T_{fluid} = -\frac{\rho}{2}\iint_s \phi \frac{\partial \phi}{\partial n}\, ds \quad (1) \qquad \text{(IV-10)}$$

where the integration is taken over all surfaces of the fluid and $\frac{\partial \phi}{\partial n}$ is the normal derivative of ϕ on the particular surface over which the integration is performed. The kinetic energy may also be obtained by a volume integration of the square of the velocity,

$$T_f = \frac{\rho}{2}\iiint_{vol.} (v_r^2 + v_\theta^2 + v_z^2)\, r dr d\theta dz \qquad \text{(IV-11-a)}$$

which, upon substitution of Eq. (IV-3) - (IV-5), becomes

$$T_f = \frac{\rho}{2}\int_0^a\int_0^{2\pi}\int_0^{\gamma L} \left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \phi}{\partial \theta}\right)^2 \quad r dr d\theta dz \qquad \text{(IV-11-b)}$$

The determination of ϕ will take place in three steps using the principle of superposition. Three separate potentials, ϕ_1, ϕ_2, and ϕ_3 will be obtained such that

$$\phi = \phi_1 + \phi_2 + \phi_3 \qquad \text{(IV-12)}$$

will satisfy the boundary value problem.

The various potential functions $\phi_k(r, \theta, z, t)$ of Eq. (IV-12) can be expressed as a summation of their respective components in the n modes:

$$\phi_k = \sum_{n=1}^{\infty} \phi_{kn}(r, \theta, z, t). \qquad \text{(IV-13)}$$

(1) For example, "Hydrodynamics" by H. Lamb, Dover Publications, Sixth Edition, 1932, pp.46, Eq. (4).

In the analysis that follows, the expressions for ϕ_{kn} will be determined and the virtual mass m_{vn} in each mode will be evaluated.

Expressions for ζ_{ni}, the displacement of the fluid surface will also be presented. It is convenient to consider the problem for the following three cases: a) $n \neq 0$ - Three Constant Approximation, b) $n = 0$ and c) $n \neq 0$ - Five Constant Approximation.

(a) $n \neq 0$ - Three Constant Approximation

Consider the potential function ϕ_{1n}

$$\phi_{1n} = - B_n \left[\frac{1}{2} + \frac{z}{\gamma L} \right] \frac{r^n}{na^{n-1}} \cos n\theta \quad \dot{q}_n(t) \tag{IV-14}$$

which satisifes the equation $\nabla^2 \phi_{1n} = 0$. The component of velocity in the radial direction is

$$v_{r1} = B_n \left[\frac{1}{2} + \frac{z}{\gamma L} \right] \left(\frac{r}{a}\right)^{n-1} \cos n\theta \; \dot{q}_n(t) \quad . \tag{IV-15}$$

Equating the radial velocity of the shell to the radial velocity of the fluid at $r = a$, Eq. (IV-7) may be used to evaluate B_n. Defining the generalized coordinate of the n^{th} shell mode as $q_n(t)$, the shell displacements are given by the relations

$$u = u(z, \theta) q_n(t) \qquad \text{(IV-16) (a)}$$

$$v = v(z, \theta) q_n(t) \qquad \text{(b)}$$

$$w = w(z, \theta) q_n(t) \qquad \text{(c)}$$

where $u(z, \theta)$, $v(z, \theta)$ and $w(z, \theta)$ are defined by Eq. (I-53)-(I-55). The value of $\dot{w}$ is obtained by differentiating Eq. (IV-16-c) and shifting the axis of z coordinates to that of Fig. (IV-1):

$$\dot{w} = - C_n \frac{\gamma L}{a} \left[\frac{1}{2} + \frac{z}{\gamma L} \right] \cos n\theta \; \dot{q}_n(t) \; ^{(2)} . \tag{IV-17}$$

Substituting Eq. (IV-15) and (IV-17) into (IV-7), the value of B_n is $B_n = - C_n \frac{\gamma L}{a}$, and the velocity components due to ϕ_{1n} becomes

$$v_{r1} = - C_n \frac{\gamma L}{a} \left[\frac{1}{2} + \frac{z}{\gamma L} \right] \left(\frac{r}{a}\right)^{n-1} \cos n\theta \; \dot{q}_n(t) \tag{IV-18}$$

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(2) The minus sign in Eq. (IV-17) is required to refer the shell velocity to a "positive velocity in the positive r direction" convention used in Eq. (IV-2) and (IV-7).

$$v_{\theta 1} = C_n \frac{\gamma L}{a} \left[\frac{1}{2} + \frac{z}{\gamma L} \right] \left(\frac{r}{a}\right)^{n-1} \sin n\theta \dot{q}_n(t) \qquad \text{(IV-19)}$$

$$v_{z1} = -C_n \frac{\gamma L}{a} \left(\frac{r^n}{\gamma L n a^{n-1}}\right) \cos n\theta \dot{q}_n(t) \qquad \text{(IV-20)}$$

The fluid potential ϕ_{1n} gives a longitudinal velocity component v_{z1} which is independent of z. In order to satisfy the boundary condition of Eq. (IV-8) a second function ϕ_{2n} must be added to ϕ_{1n}.

The velocity potential $\phi_{2n}(r, \theta, z, t)$ is chosen so that $\nabla^2\phi_{2n} = 0$ and that in conjunction with ϕ_{1n}, the boundary condition of Eq. (IV-8) is satisfied. At the same time, the radial velocity V_{r2} is chosen so that

$$v_{r2}\Big]_{r=a} = -\frac{\partial \phi_2}{\partial r}\Big]_{r=a} = 0 \qquad \text{(IV-21)}$$

since v_{r1} already satisfies the boundary condition of Eq. (IV-7). The form of ϕ_2 is taken as

$$\phi_{2n} = \sum_{i=1}^{\infty} C_{ni} \sinh \left(\frac{\alpha_{ni} z}{a}\right) J_n\left(\frac{\alpha_{ni} r}{a}\right) \cos n\theta \dot{q}_n(t) \qquad \text{(IV-22)}$$

where, in order to satisfy Eq. (IV-21), the α_{ni} are the zeros of the derivative of the Bessel Function of order n, i.e.

$$J_n'(\alpha_{ni}) = 0 \qquad \text{(IV-23)}$$

The component of velocity in the longitudinal direction is

$$V_{z_2} = -\sum_{i=1}^{\infty} C_{ni} \frac{\alpha_{ni}}{a} \cosh\left(\alpha_{ni}\frac{z}{a}\right) J_n\left(\alpha_{ni}\frac{r}{a}\right) \cos n\theta \, \dot{q}_n(t) \qquad \text{(IV-24)}$$

The boundary condition of Eq. (IV-8) may now be written as

$$\left[V_{z1} + V_{z2}\right]_{z = -\frac{\gamma L}{2}} = 0 \qquad \text{(IV-25)}$$

and upon substitution of Eq. (IV-20) and (IV-24), Eq. (IV-25) becomes

$$-\sum_{i=1}^{\infty} C_{ni}\frac{\alpha_{ni}}{a}\cosh\left(\frac{\alpha_{ni}\gamma L}{2a}\right) J_n\left(\frac{\alpha_{ni} r}{a}\right) = \left[C_n \frac{\gamma L}{a}\right]\frac{r^n}{\gamma L n a^{n-1}} \qquad \text{(IV-26)}$$

The constant C_{ni} can be evaluated using the orthogonality property of Bessel Functions. Multiplying both sides of Eq. (IV-26) by the function $J_n(\frac{\alpha_{nj} r}{a})$ where α_{nj} is any one of the α_{ni} roots of $J_n'(\alpha_{ni}) = 0$, and integrating with respect to r, one obtains

$$-\sum_{i=1}^{\infty} C_{ni}\frac{\alpha_{ni}}{a}\cosh\left(\frac{\alpha_{ni}\gamma L}{2a}\right)\int_0^a J_n\left(\frac{\alpha_{ni} r}{a}\right) J_n\left(\frac{\alpha_{nj} r}{a}\right) r\,dr =$$
$$\left[C_n \frac{\gamma L}{a}\right]\left(\frac{1}{\gamma L n a^{n-1}}\right)\int_0^a r^{n+1} J_n\left(\frac{\alpha_{nj} r}{a}\right) dr \qquad \text{(IV-27)}$$

The orthogonality relation for the Bessel functions[3] is

(3) See, for example, McLachlan, N.W., "Bessel Functions for Engineers", Oxford University Press, 1934, Pg. 160, Eq. (47) and (48).

$$\int_0^a J_n\left(\frac{\alpha_{ni}r}{a}\right) J_n\left(\frac{\alpha_{nj}r}{a}\right) r\,dr = \begin{cases} 0 & i \neq j \\ \frac{a^2}{2}\left(1-\frac{n^2}{\alpha_{ni}^2}\right) J_n^2(\alpha_{ni}) & i = j \end{cases} \qquad \text{(IV-28)}$$

Also used will be the relation

$$\int_0^a r^{n+1} J_n\left(\frac{\alpha_{ni}r}{a}\right) dr = \frac{na^{n+2}}{\alpha_{ni}^2} J_n(\alpha_{ni}) \qquad \text{(IV-29)}$$

which is valid for $J_n'(\alpha_{ni}) = 0$.

Using Eq. (IV-28) and (IV-29), the constant C_{ni} is evaluated from Eq. IV-27):

$$C_{ni} = \frac{-\left[C_n \frac{\gamma L}{a}\right] 2a^2}{\gamma L \alpha_{ni} \cosh\left(\frac{\alpha_{ni}\gamma L}{2a}\right) J_n(\alpha_{ni})(\alpha_{ni}^2 - n^2)} \qquad \text{(IV-30)}$$

The velocity components due to the function ϕ_2 become:

$$V_{r2} = -\sum_{i=1}^{\infty} C_{ni} \sinh\left(\frac{\alpha_{ni}z}{a}\right) J_n'\left(\frac{\alpha_{ni}r}{a}\right) \frac{\alpha_{ni}}{a} \cos n\theta\, \dot{q}_n(t) \qquad \text{(IV-31)}$$

$$V_{\theta 2} = \sum_{i=1}^{\infty} C_{ni} \sinh\left(\frac{\alpha_{ni}z}{a}\right) J_n\left(\frac{\alpha_{ni}r}{a}\right) \frac{n}{r} \sin n\theta\, \dot{q}_n(t) \qquad \text{(IV-32)}$$

and

V_{z2} which is given by Eq. (IV-24).

The potential function obtained by adding $\phi_{1n} + \phi_{2n}$ satisfies (Eq. IV-1) and represents a motion in which the boundary conditions of Eq. (IV-7) and Eq. (IV-8) are satisfied. Moreover, this motion maintains a plane surface at the top of the fluid, $z = \frac{\gamma L}{2}$, on which the particles have zero velocity.

To satisfy the condition of zero pressure on the free surface, a third fluid potential $\phi_{3n}(r, \theta, z, t)$ is introduced. This function represents the sloshing motions of the fluid on a partially filled tank under the conditions of zero velocity at the tank bottom and zero radial velocity at the surface of the tank $r = a$. Thus, it represents a motion in which the tank remains rigid while the fluid inside it moves. The velocity potential $\phi_{3n}(r, \theta, z, t)$ is chosen as

$$\phi_{3n} = -\sum_{i=1}^{\infty} \frac{\dot{A}_{ni}\, a \cos n\theta\, J_n\left(\frac{\alpha_{ni} r}{a}\right) \cosh\left[\frac{\alpha_{ni}}{a}\left(z + \frac{\gamma L}{2}\right)\right]}{\alpha_{ni}\, J_n(\alpha_{ni}) \sinh\left(\frac{\alpha_{ni}\gamma L}{a}\right)} \qquad (IV\text{-}33)$$

where the values of α_{ni} are again the zeros of the derivative of the Bessel Function of order n as given by Eq. (IV-23). The coordinates $A_{ni}(t)$ could be considered as the generalized coordinates of the sloshing modes of the fluid which are to be superimposed on the motions due to the tank displacements given by the potential functions $\phi_{1n} + \phi_{2n}$. The velocity components due to the function ϕ_{3n} are

$$V_{r3} = \sum_{i=1}^{\infty} \frac{\dot{A}_{ni} \cos n\theta\, J_n'\left(\frac{\alpha_{ni} r}{a}\right) \cosh\left[\frac{\alpha_{ni}}{a}\left(z + \frac{\gamma L}{2}\right)\right]}{J_n(\alpha_{ni}) \sinh\left(\frac{\alpha_{ni}\gamma L}{a}\right)} \qquad (IV\text{-}34)$$

$$v_{\theta 3} = -\sum_{i=1}^{\infty} \frac{\dot{A}_{ni}\, n a \sin n\theta\, J_n\left(\frac{\alpha_{ni} r}{a}\right) \cosh\left[\frac{\alpha_{ni}}{a}\left(z + \frac{\gamma L}{2}\right)\right]}{r \alpha_{ni} J_n(\alpha_{ni}) \sinh\left(\alpha_{ni} \frac{\gamma L}{a}\right)} \qquad \text{(IV-35)}$$

and

$$v_{z3} = \sum_{i=1}^{\infty} \frac{\dot{A}_{ni} \cos n\theta\, J_n\left(\frac{\alpha_{ni} r}{a}\right) \sinh\left[\frac{\alpha_{ni}}{a}\left(z + \frac{\gamma L}{2}\right)\right]}{J_n(\alpha_{ni}) \sinh\left(\alpha_{ni} \frac{\gamma L}{a}\right)} \qquad \text{(IV-36)}$$

Eq. (IV-34) and (IV-36) show that $\left.v_{r3}\right]_{r=a} = 0$ and that $\left.v_{z3}\right]_{z = -\frac{\gamma L}{2}} = 0$. The coordinates $A_{ni}(t)$ are as yet undetermined but may be expressed in terms of the $q_n(t)$, the generalized coordinates of the shell motion. This is done by utilization of the condition that the fluid surface at $z = \frac{\gamma L}{2}$ must be a free surface with zero pressure.

To find this relation, consider the general fluid motion governed by the potential function ϕ of Eq. (IV-13) so that the velocity of the fluid in any direction s is given by

$$v_2 = v_{s1} + v_{s2} + v_{s3} \qquad \text{(IV-37)}$$

The kinetic energy of the fluid may be determined from Eq. (IV-10), which upon substitution of the appropriate values of ϕ_n and its space derivatives on the surfaces $r = a$ and $z = \pm \frac{\gamma L}{2}$ becomes:

$$T_{fluid} = \frac{\rho}{2}\int_0^{2\pi}\int_{-\frac{\gamma L}{2}}^{+\frac{\gamma L}{2}}\left[\phi_{1n}+\phi_{2n}+\phi_{3n}\right]_{r=a}\left(\frac{\partial \phi_{1n}}{\partial r}\right)_{r=a} a\,d\theta\,dz$$

$$+\frac{\rho}{2}\int_0^{2\pi}\int_0^{a}\left[\phi_{1n}+\phi_{2n}+\phi_{3n}\right]_{z=\frac{\gamma L}{2}}\left(\frac{\partial \phi_{3n}}{\partial z}\right)_{z=\frac{\gamma L}{2}} r\,dr\,d\theta \qquad (IV-38)$$

It may be noted that $\frac{\partial}{\partial z}\left[\phi_{1n}+\phi_{2n}\right]_{z=\frac{\gamma L}{2}} = 0$ and that $\frac{\partial\phi_2}{\partial r} = \frac{\partial\phi_3}{\partial r} = 0$ on the surface $r = a$.

Substituting the values of ϕ_n and its space derivatives into Eq. (IV-38), and performing the required integrations, the kinetic energy of the fluid on the nth mode becomes:

$$T_{n\,fluid} = \frac{\pi\rho a^2\gamma L}{6n}\left[\frac{C_n^2\gamma^2L^2}{a^2}\right]\left\{1-\sum_{l=1}^{\infty}\frac{6a^2n\left[1-\frac{2a}{\alpha_{nl}\gamma L}\tanh\left(\frac{\alpha_{nl}\gamma L}{2a}\right)\right]}{\alpha_{nl}^2\gamma^2L^2(\alpha_{nl}^2-n^2)}\right\}\dot{q}_n^2(t) -$$

$$-\frac{\pi\rho a^3}{2}\left[C_n\frac{\gamma L}{a}\right]\sum_{l=1}^{\infty}\frac{1}{\alpha_{nl}^2}\left\{2-\frac{1}{\tanh\left(\alpha_{nl}\frac{\gamma L}{a}\right)\left(\alpha_{nl}\frac{\gamma L}{a}\right)}+\frac{1}{\sinh\left(\alpha_{nl}\frac{\gamma L}{a}\right)\left(\alpha_{nl}\frac{\gamma L}{a}\right)}-\right.$$

$$\left.-\frac{\tanh\left(\alpha_{nl}\frac{\gamma L}{2a}\right)}{\alpha_{nl}\frac{\gamma L}{a}}\right\}\dot{q}_n(t)\dot{A}_{nl}(t)+\frac{\rho\pi a^3}{4}\sum_{l=1}^{\infty}\frac{(\alpha_{nl}^2-n^2)}{\alpha_{nl}^3\tanh\left(\alpha_{nl}\frac{\gamma L}{a}\right)}\dot{A}_{nl}^2(t) \qquad (IV-39)$$

Eq. (IV-39) can be written as

$$T_{n\,fluid} = D\dot{q}_n^2+\sum_{l=1}^{\infty}D_{1l}\dot{q}_n^2+\sum_{l=1}^{\infty}D_{2l}\dot{q}_n\dot{A}_{nl}+\sum_{l=1}^{\infty}D_{3l}\dot{A}_{nl}^2 \qquad (IV-40)$$

where

$$D = \frac{\pi \rho a^2 \gamma L}{6n}\left[C_n^2 \frac{\gamma^2 L^2}{a^2}\right] \tag{IV-41}$$

$$D_{1i} = -D\left\{\frac{6a^2 n\left[1 - \frac{2a}{\alpha_{ni}\gamma L}\tanh\left(\frac{\alpha_{ni}\gamma L}{2a}\right)\right]}{\alpha_{ni}^2 \gamma^2 L^2 (\alpha_{ni}^2 - n^2)}\right\} \tag{IV-42}$$

$$D_{2i} = -\frac{\pi \rho a^3}{2}\left[C_n \frac{\gamma L}{a}\right]\frac{1}{\alpha_{ni}^2}\left\{2 - \frac{1}{\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)\left(\alpha_{ni}\frac{\gamma L}{a}\right)} + \frac{1}{\sinh\left(\alpha_{ni}\frac{\gamma L}{a}\right)\left(\alpha_{ni}\frac{\gamma L}{a}\right)} - \frac{\tanh\left(\alpha_{ni}\frac{\gamma L}{2a}\right)}{\alpha_{ni}\frac{\gamma L}{a}}\right\} \tag{IV-43}$$

and

$$D_{3i} = \frac{\rho \pi a^3}{4}\frac{(\alpha_{ni}^2 - n^2)}{\alpha_{ni}^3 \tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} \tag{IV-44}$$

Let the kinetic energy of the shell in the n^{th} mode be given as

$$T_{nshell} = \frac{1}{2} M_n \dot{q}_n^2 \tag{IV-45}$$

and the potential energy of the shell be

$$V_{nshell} = \frac{1}{2} \bar{K}_n q_n^2 \tag{IV-46}$$

The equations of motion governing the generalized coordinates $q_n(t)$ and $A_{ni}(t)$ may be obtained by the use of Lagranges equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial V}{\partial q_k} = Q_k \tag{IV-47}$$

where T is the total kinetic energy of the system in the k^{th} mode and is given by the sum of Eq. (IV-40) and (IV-45); V is the potential energy of the system and is given by Eq. (IV-46); and Q_k is the generalized force in the mode k due to any external loading and to body forces such as the force of gravity.

Noting that the pressure on the fluid surface $z = \frac{\gamma L}{2}$ is zero and neglecting the effect of the gravity forces[4], the generalized force Q_k is zero for free vibrations and the Lagrange equations on the generalized coordinates q_n and A_{ni} become:

$$(2D+M_n)\ddot{q}_n + \left(2\sum_{i=1}^{\infty} D_{1i}\right)\ddot{q}_n + \sum_{i=1}^{\infty} D_{2i}\ddot{A}_{ni} + \bar{K}_n q_n = 0 \qquad \text{(IV-48)}$$

$$D_{2i}\ddot{q}_n + 2D_{3i}\ddot{A}_{ni} = 0 \qquad (i = 1, 2, 3 \cdots \infty) \qquad \text{(IV-49)}$$

where Eq. (IV-49) holds for each value of i.

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(4) If gravity is included in the analysis, there is a contribution to the generalized force term, Q_k, which contains terms A_{ni}. The analysis of this complete system gives rise to two sets of modes, one of which is very close to the tank modes q_n which are obtained when gravity is neglected. The second set of modes are very nearly the sloshing modes of the fluid in a rigid tank in the presence of gravity. These modes have extremely long periods as compared to the q_n modes and give practically no contribution to the displacements and stresses in the tank. The gravity sloshing modes are required for the analysis of possible fluid spilling from the tank and will be considered in a later section of the report. The modes q_n obtained by neglecting gravity will thus be used to determine the stresses and displacements in the shell.

Solving Eq. (IV-49) for $\ddot{A}_{ni}$,

$$\ddot{A}_{ni} = -\frac{D_{2i}}{2D_{3i}}\ddot{q}_n \qquad (i = 1, 2, 3, \cdots \infty) \qquad (IV-50)$$

and substituting Eq. (IV-50) into Eq. (IV-48), we obtain the equation on the generalized coordinate $q_n(t)$,

$$\left[2D + M_n + \sum_{i=1}^{\infty}\left(2D_{1i} - \frac{D_{2i}^2}{2D_{3i}}\right)\right]\ddot{q}_n + \bar{K}_n q_n = 0 \qquad (IV-51)$$

Equation (IV-51) implies that the kinetic energy of the fluid, Eq. (IV-39), (IV-40) can then be written in terms of the generalized coordinate $q_n(t)$ only:

$$T_{n\,fluid} = \bar{C}_n \dot{q}_n^2(t) \qquad (IV-52)$$

where the value of $\bar{C}_n$ is given by

$$\bar{C}_n = D + \sum_{i=1}^{\infty}\left(D_{1i} - \frac{D_{2i}^2}{4D_{3i}}\right) \qquad (IV-53)$$

Substituting the approximate constants into Eq. (IV-52) and Eq. (IV-53), the kinetic energy of the fluid in the n^{th} mode becomes:

$$T_{n\,fluid} = \left[C_n^2\frac{\gamma^2L^2}{a^2}\right]\pi\rho a^2\gamma L\left\{\frac{1}{6n}\left[1 - \sum_{i=1}^{\infty}\frac{6a^2n\left[1 - \frac{2a}{\alpha_{ni}\gamma L}\tanh\left(\alpha_{ni}\frac{\gamma L}{2a}\right)\right]}{\alpha_{ni}^2\gamma^2L^2(\alpha_{ni}^2 - n^2)}\right] - \right.$$
$$- \sum_{i=1}^{\infty}\frac{\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}{\alpha_{ni}\frac{\gamma L}{a}(\alpha_{ni}^2 - n^2)}\left[1 - \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} + \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\sinh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}\right.$$
$$\left.\left. - \frac{\tanh\left(\frac{\alpha_{ni}\gamma L}{2a}\right)}{2\alpha_{ni}\frac{\gamma L}{a}}\right]^2\right\}\dot{q}_n^2(t) \qquad (IV-54)$$

The "virtual mass" of the fluid in the n^{th} mode, m_{vn}, is defined so that the kinetic energy of the fluid in the n^{th} mode is given by

$$T_{n\,fluid} = \frac{m_{vn}}{2}\int_0^{2\pi}\int_0^{\gamma L} \dot{w}_n^2(z,\theta,t)\, a\, d\theta\, dz \qquad \text{(IV-55)}$$

where $w_n(z, \theta, t)$, the radial displacement of the shell is given by Eq. (IV-16c). The total kinetic energy of the partially filled shell plus the fluid in the n^{th} mode becomes

$$T_n = \frac{m}{2}\int_0^{2\pi}\int_0^{L}\left(\dot{u}_n^2 + \dot{v}_n^2 + \dot{w}_n^2\right) a\, d\theta\, dz + \frac{m_{vn}}{2}\int_0^{2\pi}\int_0^{\gamma L} \dot{w}_n^2\, a\, d\theta\, dz\,. \qquad \text{(IV-56)}$$

The virtual mass of the fluid can thus be considered to be an additional tank mass moving only in the radial direction.

Substituting Eq. (IV-16c) into Eq. (IV-55) and equating the result to Eq. (IV-54), the virtual mass of the fluid in the n^{th} mode, m_{vn}, becomes:

$$m_{vn} = \rho a\left\{\frac{1}{n}\left[1 - \sum_{i=1}^{\infty}\frac{6a^2 n\left[1 - \frac{2a}{\alpha_{ni}\gamma L}\tanh\left(\frac{\alpha_{ni}\gamma L}{2a}\right)\right]}{\alpha_{ni}^2\gamma^2 L^2\left(\alpha_{ni}^2 - n^2\right)}\right] - \sum_{i=1}^{\infty}\frac{6\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}{\alpha_{ni}\frac{\gamma L}{a}\left(\alpha_{ni}^2 - n^2\right)}\right.$$

$$\left.\left[1 - \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} + \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\sinh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} - \frac{\tanh\left(\alpha_{ni}\frac{\gamma L}{2a}\right)}{2\alpha_{ni}\frac{\gamma L}{a}}\right]^2\right\} \qquad \text{(IV-57)}$$

For convenience Eq. (IV-57) may be written as

$$m_{vn} = \epsilon_n \rho a \qquad \text{(IV-58)}$$

where ϵ_n is the quantity inside the braces. Figure (IV-2) shows the variation of ϵ_n with the mode n for the various fillings denoted by γ = 0.5, 0.75, 0.9, and 1.0. The computations were made for a tank for which L/a = 0.8 and include terms up to and including i = 3 in the summation in ϵ_n.

The motion of the fluid surface at $z = + \frac{\gamma L}{2}$ in each mode n, is required for the analysis of the response of the partially filled tank to dynamic loading. This analysis is given in Chapter VI of this Report. These motions can be obtained by an integration with respect to time of the longitudinal velocity V_z on the fluid surface $z = + \frac{\gamma L}{2}$. Using Eq. (IV-37) and noting that

$$\left. V_{z1} + V_{z2} \right]_{z = \frac{\gamma L}{2}} = 0 \qquad \text{(IV-59)}$$

the velocity of the fluid on the surface $z = + \frac{\gamma L}{2}$ is given by V_{z3}. Using Eq. (IV-36) and Eq. (IV-50), the velocity becomes:

$$\left. V_z \right]_{z=+\frac{\gamma L}{2}} = \sum_{i=1}^{\infty} \left[C_n \frac{\gamma L}{a} \right] \frac{2\alpha_{ni} \tanh\left(\alpha_{ni} \frac{\gamma L}{a}\right)}{\left(\alpha_{ni}^2 - n^2\right)} \left[1 - \frac{1}{2\alpha_{ni} \frac{\gamma L}{a}} \left(\frac{1}{\tanh\left(\alpha_{ni} \frac{\gamma L}{a}\right)} - \frac{1}{\sinh\left(\alpha_{ni} \frac{\gamma L}{a}\right)} + \tanh\left(\alpha_{ni} \frac{\gamma L}{2a}\right) \right) \right] \frac{J_n\left(\alpha_{ni} \frac{r}{a}\right)}{J_n(\alpha_{ni})} \cos n\theta \; g_n(t) \qquad \text{(IV-60)}$$

Integrating with respect to time and noting that the initial displacement of the fluid is zero, the fluid displacement in the n^{th} mode (Measured positive upward), $\zeta_n(r, \theta, t)$ is given by the relation

$$\zeta_n(r, \theta, t) = \sum_{i=1}^{\infty} \zeta_{ni}(r, \theta, t) \qquad \text{(IV-61a)}$$

where

$$\zeta_{ni}(r,\theta,t) = \left[C_n \frac{\gamma L}{a}\right] \frac{2\alpha_{ni} \tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}{(\alpha_{ni}^2 - n^2)} \left[1 - \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}}\left(\frac{1}{\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} - \right.\right.$$

$$\left.\left. - \frac{1}{\sinh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} + \tanh\left(\alpha_{ni}\frac{\gamma L}{2a}\right)\right)\right] \frac{J_n\left(\alpha_{ni}\frac{r}{a}\right)}{J_n(\alpha_{ni})} \cos n\theta \, q_n(t) \qquad \text{(IV-61b)}$$

Rewriting Eq. (IV-61b) as

$$\zeta_{ni}(r,\theta,t) = \frac{C_n \gamma L}{a} \beta_{ni} J_n\left(\frac{\alpha_{ni} r}{a}\right) \cos n\theta \, q_n(t) \qquad \text{(IV-62a)}$$

where

$$\beta_{ni} = \frac{2\alpha_{ni}\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}{J_n(\alpha_{ni})(\alpha_{ni}^2 - n^2)} \left[1 - \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}}\left(\frac{1}{\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} - \frac{1}{\sinh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} + \right.\right.$$

$$\left.\left. + \tanh\left(\alpha_{ni}\frac{\gamma L}{2a}\right)\right)\right] \qquad \text{(IV-62b)}$$

and considering a tank with $L/a = 0.8$, the values of the coefficients β_{ni} are given in Figures (IV-3)-(IV-8) for the modes $n = 1$-6 and $i = 1, 2, 3$. In each case, the coefficient is given for tank fillings ranging from $\gamma = 0.50$ to $\gamma = 1.00$. Table (IV-1) gives the corresponding values of the roots α_{ni} for use in Eq. (IV-62a).

b) n = 0

Consider the potential function ϕ_1 (r, z, t)

$$\phi_1 = \frac{1}{a}\left[\frac{r^2}{2} - (z + \frac{\gamma L}{2})^2\right] \dot{q}_o(t) \qquad \text{(IV-63)}$$

which satisfies the equation $\nabla^2\phi_1 = 0$. The component of velocity in the radial direction is

$$v_{r1} = -\frac{r}{a} \dot{q}_o(t) \quad . \qquad \text{(IV-64)}$$

For application in the forced vibration analysis of Section (VI), a z-independent radial displacement of the shell, $w_o = q_o(t)$, will be considered, where $q_o(t)$ is the generalized coordinate of the shell mode n = 0.

Noting that the radial velocity of the shell at r = a is given by

$$\dot{w}_o = -\dot{q}_o^{(5)} \qquad \text{(IV-65)}$$

the boundary condition of Eq. (IV-7) is satisfied by the potential function ϕ_1 (r, z, t).

The velocity components in the tangential and longitudinal directions are respectively

$$v_{\theta_1} = 0 \qquad \text{(IV-66)}$$

$$v_{z_1} = +\frac{2}{a}(z + \frac{\gamma L}{2}) \dot{q}_o(t) \cdot \qquad \text{(IV-67)}$$

The fluid potential ϕ_1 gives a longitudinal velocity component $v_z = 0$ at the bottom of the tank, $z = -\frac{\gamma L}{2}$, thus satisfying the boundary condition

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(5) See footnote (2) on Page (IV-5)

of Eq. (IV-8).

To satisfy the condition of zero pressure on the free surface, a second fluid potential $\phi_2(r, z, t)$ is introduced. This function represents the sloshing motion of the fluid in a partially filled rigid tank. The velocity potential ϕ_2 is chosen as

$$\phi_2(r,z,t) = -\sum_{i=1}^{\infty} \frac{\dot{A}_{oi}\, a\, J_o\left(\alpha_{oi}\frac{r}{a}\right)\cosh\left[\frac{\alpha_{oi}}{a}\left(z+\frac{\gamma L}{2}\right)\right]}{\alpha_{oi}\, J_o(\alpha_{oi})\sinh\left(\alpha_{oi}\frac{\gamma L}{a}\right)} \qquad \text{(IV-68)}$$

where the values of α_{ni} are the zeros of the derivative of the Bessel Functions of order zero,

$$J_o'(\alpha_{oi}) = -J_1(\alpha_{oi}) = 0 \qquad \text{(IV-69)}$$

The velocity components due to the function ϕ_2 are

$$v_{r_2} = -\sum_{i=1}^{\infty} \frac{\dot{A}_{oi}\, J_1\left(\alpha_{oi}\frac{r}{a}\right)\cosh\left[\frac{\alpha_{oi}}{a}\left(z+\frac{\gamma L}{2}\right)\right]}{J_o(\alpha_{oi})\sinh\left(\alpha_{oi}\frac{\gamma L}{a}\right)} \qquad \text{(IV-70)}$$

$$v_{\theta_2} = 0 \qquad \text{(IV-71)}$$

$$v_{z_2} = \sum_{i=1}^{\infty} \frac{\dot{A}_{oi}\, J_o\left(\alpha_{oi}\frac{r}{a}\right)\sinh\left[\frac{\alpha_{oi}}{a}\left(z+\frac{\gamma L}{2}\right)\right]}{J_o(\alpha_{oi})\sinh\left(\alpha_{oi}\frac{\gamma L}{a}\right)} \qquad \text{(IV-72)}$$

Eq. (IV-70) and (IV-72) show that $v_{r_2}\Big]_{r=a} = 0$ and that $v_{z_2}\Big]_{z=-\frac{\gamma L}{2}} = 0$. The corodinates $A_{oi}(t)$ may be expressed in terms of the generalized coordinate of the shell, $q_o(t)$ in a manner similar to the case where $n \neq 0$.

To find this relation, consider the general fluid motion governed by the potential function $\phi = \phi_1 + \phi_2$ so that the velocity of the fluid in any direction s is given by $v_s = v_{s1} + v_{s2}$. The kinetic energy of the fluid may be determined from Eq. (IV-11),

$$T_{o\,fluid} = \frac{\rho}{2}\int_0^{2\pi}\int_0^{\gamma L}\int_0^{a}\left\{\left[\frac{-\partial}{\partial r}(\phi_1+\phi_2)\right]^2 + \left[\frac{-\partial}{\partial z}(\phi_1+\phi_2)\right]^2\right\} r\,dr\,d\theta\,dz \tag{IV-73}$$

which upon substitution of the appropriate values of the space derivatives of ϕ becomes:

$$T_{o\,fluid} = \pi\rho\left[\frac{a^2\gamma L}{4} + \frac{2}{3}\gamma^3L^3\right]\dot{q}_o^{\,2} - \sum_{i=1}^{\infty}\frac{2\pi\rho a^3}{\alpha_{oi}^2}\dot{q}_o\dot{A}_{oi} + \sum_{i=1}^{\infty}\frac{\pi\rho a^3\,\dot{A}_{oi}^{\,2}}{2\alpha_{oi}\tanh\left(\frac{\alpha_{oi}\gamma L}{a}\right)} \tag{IV-74}$$

Eq. (IV-74) may be written as

$$T_{o\,fluid} = D_o\,\dot{q}_o^{\,2} + \sum_{i=1}^{\infty} D_{o2}\,\dot{q}_o\dot{A}_{oi} + \sum_{i=1}^{\infty} D_{o3}\,\dot{A}_{oi}^{\,2} \tag{IV-75}$$

where

$$D_o = \pi\rho\left[\frac{a^2\gamma L}{4} + \frac{2}{3}\gamma^3L^3\right] \tag{IV-76}$$

$$D_{o2} = -\frac{2\pi\rho a^3}{\alpha_{oi}^2} \tag{IV-77}$$

and

$$D_{o3} = \frac{\pi\rho a^3}{2\alpha_{oi}\tanh\left(\frac{\alpha_{oi}\gamma L}{a}\right)} \tag{IV-78}$$

Let the kinetic energy of the shell in the zero mode be

$$T_{oshell} = \frac{1}{2} \bar{M}_o \dot{q}_o^2 \qquad \text{(IV-79)}$$

and the potential energy of the shell be

$$V_{oshell} = \frac{1}{2} \bar{K}_o q_o^2 \quad . \qquad \text{(IV-80)}$$

Proceeding in a similar manner to the case where $n \neq 0$, the Lagrangian equations of motion on the generalized coordinates $q_o(t)$ and $A_{oi}(t)$ become:

$$\left[\bar{M}_o + 2D_o\right] \ddot{q}_o + \sum_{i=1}^{\infty} D_{o2} \ddot{A}_{oi} + \bar{K}_o q_o = 0 \qquad \text{(IV-81)}$$

$$D_{o2} \ddot{q}_o + 2D_{o3} \ddot{A}_{oi} = 0 \qquad (i = 1, 2, 3, \ldots \infty) \qquad \text{(IV-82)}$$

Eq. (IV-81) and (IV-82) satisfy the condition that the pressure on the fluid surface $z = \frac{\gamma L}{2} = 0$.

It may also be noted that the effect of gravity forces has again been neglected.

Solving Eq. (IV-82) for A_{oi},

$$A_{oi} = - \frac{D_{o2}}{2D_{o3}} q_o \qquad \text{(IV-83)}$$

and substituting this value into Eq. (IV-81), the equation on the generalized coordinate $q_o(t)$ becomes:

$$\left[2D_o + M_o - \sum_{i=1}^{\infty} \frac{D_{o2}^2}{2D_{o3}}\right] \ddot{q}_o + \bar{K}_o q_o = 0 \qquad \text{(IV-84)}$$

Equation (IV-84) implies that the kinetic energy of the fluid can be written in terms of the generalized coordinate $q_o(t)$ only,

$$T_{o\,fluid} = \overline{C}_o\, \dot{q}_o^2\,(t) \qquad \text{(IV-85)}$$

where the value of $\overline{C}_o$ is given by

$$\overline{C}_o = D_o - \sum_{i=1}^{\infty} \frac{D_{o2}^2}{4D_{o3}} \qquad \text{(IV-86)}$$

Substituting the appropriate constants into Eq. (IV-85) and Eq. (IV-86), the kinetic energy of the fluid in the 0 mode becomes

$$T_{o\,fluid} = \pi\rho a^2\left[\frac{\gamma L}{4} + \frac{2}{3}\frac{\gamma^3 L^3}{a^2} - \sum_{l=1}^{\infty}\frac{2a}{\alpha_{ol}^3}\tanh\left(\alpha_{ol}\frac{\gamma L}{a}\right)\right]\dot{q}_o^{\,2}(t) \qquad \text{(IV-87)}$$

The "virtual mass" of the fluid in the zero mode, m_{vo}, is defined so that the kinetic energy of the fluid is given by

$$T_{o\,fluid} = \frac{m_{vo}}{2}\int_0^{2\pi}\int_0^{\gamma L}\dot{w}_o^2\, u\, du\, d\theta\, dz \qquad \text{(IV-88)}$$

and $\dot{w}_o$ is given by Eq. (IV-65). Substituting this value of w_o into Eq. (IV-88) and equating Eq. (IV-87) and Eq. (IV-88), the virtual mass of the fluid, m_{vo}, becomes

$$m_{vo} = \rho a\left[\frac{L}{4} + \frac{2\gamma^2 L^2}{3a^2} - \sum_{l=1}^{\infty}\frac{2a\tanh\left(\alpha_{ol}\frac{\gamma L}{a}\right)}{\alpha_{ol}^3\,\gamma L}\right] \qquad \text{(IV-89)}$$

The motion of the fluid surface at $z = +\frac{\gamma L}{2}$ in the zero mode is obtained by an integration with respect to time of the longitudinal velocity v_z of the fluid surface $z = +\frac{\gamma L}{2}$. This velocity is given by the sum of Eq. (IV-67) and Eq. (IV-72). Using the relation of Eq. (IV-83),

the velocity becomes

$$V_z\Big]_{z=+\frac{\gamma L}{2}} = \left[2\frac{\gamma L}{a} + \sum_{i=1}^{\infty}\frac{2\tanh\left(\alpha_{oi}\frac{\gamma L}{a}\right)J_0\left(\alpha_{oi}\frac{r}{a}\right)}{\alpha_{oi}\,J_0(\alpha_{oi})}\right]\dot{q}_o(t) \qquad \text{(IV-90)}$$

Integrating with respect to time and noting that the initial displacement of the fluid is zero, the fluid displacement at the surface $z = +\frac{L}{2}$ in the mode $n = 0$ (measured positive upward), $\zeta_o(r, t)$, is given by the relation

$$\zeta_o(r,t) = \frac{2\gamma L}{a}q_o(t) + \sum_{i=1}^{\infty}\zeta_{oi}(r,t) \qquad \text{(IV-91)}$$

where

$$\zeta_{oi}(r,t) = \frac{2\tanh\left(\alpha_{oi}\frac{\gamma L}{a}\right)J_0\left(\alpha_{oi}\frac{r}{a}\right)}{\alpha_{oi}\,J_0(\alpha_{oi})}\,q_o(t) \qquad \text{(IV-92)}$$

Writing Eq. (IV-92) as

$$\zeta_{oi}(r, t) = \beta_{oi}\,J_0\left(\frac{\alpha_{oi}r}{a}\right)q_o(t) \qquad \text{(IV-93)}$$

the values of β_{oi} are given in Fig. (IV-9) for a tank in which $\frac{L}{a} = 0.8$. The coefficients β_{oi} are given for tank fillings ranging from $\gamma = 0.50$ to 1.00 and for $i = 1$, 2 and 3.

Using Eq. (IV-58) with $n = 0$ and (IV-89), the values of ϵ_o for a tank in which $L/a = 0.8$ are shown in Fig. (IV-2) for the fillings $\gamma = 0.5$, 0.75, 0.90 and 1.00.

c) $n \neq 0$ Five Constant Approximation

The procedure used in the present section for the determination of the virtual mass and the fluid displacements is exactly analagous to the procedure used for the three constant approximation of Section (IV-a).

Consider the potential function ϕ_{1n}

$$\phi_{1n} = - B_n \left[G_n + \frac{z}{\gamma L} \right] \frac{r^n}{na^{n-1}} \cos n\theta \ \dot{q}_n(t) \qquad \text{(IV-94)}$$

$$G_n = \frac{1}{2\gamma} (\gamma - 1) + \frac{a}{\gamma L} \frac{Y_n}{W_n} . \qquad \text{(IV-95)}$$

which satisfies the equation $\nabla^2 \phi_{1n} = 0$. The component of velocity in the radial direction is

$$v_{r1} = B_n \left[G_n + \frac{z}{\gamma L} \right] \left(\frac{r}{a}\right)^{n-1} \cos n\theta \ \dot{q}_n(t) \quad . \qquad \text{(IV-96)}$$

Equating the radial velocity of the shell to the radial velocity of the fluid at $r = a$, Eq. (IV-7) may be used to evaluate B_n. Defining the generalized coordinate of the n^{th} shell mode as $q_n(t)$, the shell displacements are given by Eq. (IV-16) where $u(z, \theta)$, $v(z,\theta)$ and $w(z,\theta)$ are defined by Eq. (I-25)-(I-27). The value of $\dot{w}$ is obtained by differentiating Eq. (I-27) and shifting the axis of z coordinates to that of Fig. (IV-1):

$$\dot{w} = - C_n \frac{\gamma L}{a} \left[G_n + \frac{z}{\gamma L} \right] \cos n\theta \ \dot{q}_n(t) \quad . \qquad \text{(IV-97)}$$

Substituting Eq. (IV-96) and (IV-97) into (IV-7), the value of B_n is

$$B_n = - \frac{C_n \gamma L}{a} \qquad \text{(IV-98)}$$

and the velocity components due to ϕ_{1n} become

$$v_{r1} = - C_n \frac{\gamma L}{a}\left[G_n + \frac{z}{\gamma L}\right] \left(\frac{r}{a}\right)^{n-1} \cos n\theta \;\; \dot{q}_n(t) \qquad \text{(IV-99)}$$

$$v_{\theta 1} = C_n \frac{\gamma L}{a}\left[G_n + \frac{z}{\gamma L}\right] \left(\frac{r}{a}\right)^{n-1} \sin n\theta \;\; \dot{q}_n(t) \qquad \text{(IV-100)}$$

$$v_{z1} = - C_n \frac{\gamma L}{a}\left[\frac{r^n}{\gamma L n a^{n-1}}\right] \cos n\theta \;\; \dot{q}_n(t) \;. \qquad \text{(IV-101)}$$

Proceding as in Section (V-a) and noting that the expression for the longitudinal velocity v_{z1}, Eq. (IV-101), is of the same form as Eq(IV-20), the potential function ϕ_{2n} and the corresponding longitudinal velocity v_{z2} are given by Eq. (IV-22) and (IV-24) respectively. Using the boundary condition, Eq. (IV-25), the expressions for C_{ni} and the velocities v_{r2} and $v_{\theta 2}$ are given by Eq. (IV-30)-(IV-32).

To satisfy the condition of zero pressure on the free surface, the fluid potential ϕ_{3n} is introduced. The expressions for ϕ_{3n} and the corresponding velocity components, v_{r3}, $v_{\theta 3}$ and v_{z3} are given by Eq. (IV-33)-(IV-36) where the coefficients $\dot{A}_{ni}$ are still to be determined. The coordinates $A_{ni}(t)$ can be expressed in terms of $q_n(t)$, the generalized coordinates of the shell motion. This is done by utilization of the condition that the fluid surface at $z = \frac{\gamma L}{2}$ must be a free surface with zero pressure.

To determine this relation, the kinetic energy of the fluid, Eq. (IV-38) is evaluated:

$$T_{n\,fluid} = \frac{\pi\rho a^2\gamma L}{6n}\left[\frac{C_n^2\gamma^2L^2}{a^2}\right]\left\{3G_n^2+\frac{1}{4}-\sum_{l=1}^{\infty}\frac{6a^2n\left[1-\frac{2a}{\alpha_{nl}\gamma L}\tanh\left(\frac{\alpha_{nl}\gamma L}{2a}\right)\right]}{\alpha_{nl}^2\gamma^2L^2(\alpha_{nl}^2-n^2)}\right\}\dot{q}_n^2(t)-$$

$$-\frac{\pi\rho a^3}{2}\left[C_n\frac{\gamma L}{a}\right]\sum_{l=1}^{\infty}\frac{1}{\alpha_{nl}^2}\left\{2G_n+1-\frac{1}{\tanh\left(\alpha_{nl}\frac{\gamma L}{a}\right)\left(\alpha_{nl}\frac{\gamma L}{a}\right)}+\frac{1}{\sinh\left(\alpha_{nl}\frac{\gamma L}{a}\right)\left(\alpha_{nl}\frac{\gamma L}{a}\right)}-\right.$$

$$\left.-\frac{\tanh\left(\alpha_{nl}\frac{\gamma L}{2a}\right)}{\alpha_{nl}\frac{\gamma L}{a}}\right\}\dot{q}_n(t)\dot{A}_{nl}(t)+\frac{\rho\pi a^3}{4}\sum_{l=1}^{\infty}\frac{(\alpha_{nl}^2-n^2)}{\alpha_{nl}^3\tanh\left(\alpha_{nl}\frac{\gamma L}{a}\right)}\dot{A}_{nl}^2(t) \qquad (IV-102)$$

Eq. (IV-102) can be written as

$$T_{n\,fluid} = \bar{D}\dot{q}_n^2+\sum_{l=1}^{\infty}D_{1l}\dot{q}_n^2+\sum_{l=1}^{\infty}\bar{D}_{2l}\dot{q}_n\dot{A}_{nl}+\sum_{l=1}^{\infty}D_{3l}\dot{A}_{nl}^2 \qquad (IV-103)$$

where

$$\bar{D}=\frac{\pi\rho a^2\gamma L}{6n}\left[C_n^2\frac{\gamma^2L^2}{a^2}\right]\left\{3G_n^2+\frac{1}{4}\right\} \qquad (IV-104)$$

$$\bar{D}_{2l}=-\frac{\pi\rho a^3}{2}\left[C_n\frac{\gamma L}{a}\right]\frac{1}{\alpha_{nl}^2}\left\{2G_n+1-\frac{1}{\tanh\left(\alpha_{nl}\frac{\gamma L}{a}\right)\left(\alpha_{nl}\frac{\gamma L}{a}\right)}+\right.$$

$$\left.+\frac{1}{\sinh\left(\alpha_{nl}\frac{\gamma L}{a}\right)\left(\alpha_{nl}\frac{\gamma L}{a}\right)}-\frac{\tanh\left(\alpha_{nl}\frac{\gamma L}{2a}\right)}{\alpha_{nl}\frac{\gamma L}{a}}\right\} \qquad (IV-105)$$

and D_{1i} and D_{3i} are given by Eq. (IV-42) and (IV-44) respectively.

Proceding as in Section (IV-a) and using Lagranges equations, the relation between the coordinates A_{ni} and q_n is given by

$$\ddot{A}_{ni}=\frac{-\bar{D}_{2i}}{2D_{3i}}\ddot{q}_n\ , \qquad (IV-106)$$

Using Eq. (IV-106), the kinetic energy of the fluid may be written in terms of the coordinate $q_n(t)$:

$$T_{n\,fluid} = \left[C_n^2\,\frac{\gamma^2 L^2}{a^2}\right]\pi\rho a^2\gamma L\left\{\frac{1}{6n}\left[3G_n^2+\frac{1}{4}-\sum_{i=1}^{\infty}\frac{6a^2 n\left[1-\frac{2a}{\alpha_{ni}\gamma L}\tanh\left(\alpha_{ni}\frac{\gamma L}{2a}\right)\right]}{\alpha_{ni}^2\gamma^2L^2\left(\alpha_{ni}^2-n^2\right)}\right]-\right.$$

$$-\sum_{i=1}^{\infty}\frac{\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}{\alpha_{ni}\frac{\gamma L}{a}\left(\alpha_{ni}^2-n^2\right)}\left[G_n+\frac{1}{2}-\frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}+\frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\sinh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}-\right.$$

$$\left.\left.-\frac{\tanh\left(\alpha_{ni}\frac{\gamma L}{2a}\right)}{2\alpha_{ni}\frac{\gamma L}{a}}\right]^2\right\}\dot{q}_n^2(t) \qquad (IV-107)$$

Substituting Eq. (IV-95) for G_n into Eq. (IV-107), the kinetic energy of the fluid in the mode "n" is:

$$T_{n\,fluid} = \frac{1}{2}\,\overline{\overline{T}}_{n\,fluid}\,\dot{q}_n^2(t) = \left[\frac{C_n^2\gamma^2L^2}{6a^2}\right]\pi\rho a^2\gamma L\left[\eta_n+\overline{\overline{\eta}}_n\frac{Y_n}{W_n}+\overline{\eta}_n\frac{Y_n^2}{W_n^2}\right]\dot{q}_n^2(t) \qquad (IV-108)$$

where the coefficients η_n, $\overline{\eta}_n$ and $\overline{\overline{\eta}}_n$ are respectively

$$\eta_n = \frac{1}{4n\gamma^2}\left(3+4\gamma^2-6\gamma\right)-\sum_{i=1}^{\infty}\frac{6a^2\left[1-\frac{2a}{\alpha_{ni}\gamma L}\tanh\left(\alpha_{ni}\frac{\gamma L}{2a}\right)\right]}{\alpha_{ni}^2\gamma^2L^2\left(\alpha_{ni}^2-n^2\right)}-$$

$$-\sum_{i=1}^{\infty}\frac{6\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}{\left(\alpha_{ni}\frac{\gamma L}{a}\right)\left(\alpha_{ni}^2-n^2\right)}\left[\frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\sinh\left(\frac{\alpha_{ni}\gamma L}{a}\right)}-\frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}-\frac{\tanh\left(\frac{\alpha_{ni}\gamma L}{2a}\right)}{2\alpha_{ni}\frac{\gamma L}{a}}+\right.$$

$$\left.+\frac{2\gamma-1}{2\gamma}\right]^2 \qquad (IV-109a)$$

$$\bar{\eta}_n = \frac{3a^2}{n\gamma^2 L^2} - \sum_{i=1}^{\infty} \frac{6a^3}{\gamma^3 L^3} \frac{\tanh(\alpha_{ni}\frac{\gamma L}{a})}{\alpha_{ni}(\alpha_{ni}^2 - n^2)} \qquad (IV-109\,b)$$

and

$$\bar{\bar{\eta}}_n = \frac{3a(\gamma-1)}{n\gamma^2 L} - \sum_{i=1}^{\infty} \frac{12a^2 \tanh(\alpha_{ni}\frac{\gamma L}{a})}{\alpha_{ni}\gamma^2 L^2(\alpha_{ni}^2 - n^2)} \left[\frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\sinh(\alpha_{ni}\frac{\gamma L}{a})} - \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}\tanh(\alpha_{ni}\frac{\gamma L}{a})} - \frac{\tanh(\alpha_{ni}\frac{\gamma L}{2a})}{2\alpha_{ni}\frac{\gamma L}{a}} + \frac{2\gamma-1}{2\gamma} \right] \qquad (IV-109C)$$

The motion of the fluid surface at $z = +\frac{\gamma L}{2}$ in each mode n, is required for the analysis of the response of the partially filled tank to dynamic loading. These motions can be obtained by an integration with respect to time of the longitudinal velocity v_{z3} on the fluid surface $z = +\frac{\gamma L}{2}$. Proceeding as in Section (IV-a), the fluid displacement in the n^{th} mode (positive upward), $\zeta_n(r, \theta, t)$ is given by the relation

$$\zeta_n(r, \theta, t) = \sum_{i=1}^{\infty} \zeta_{ni}(r, \theta, t) \qquad (IV-110)$$

where

$$\zeta_{ni}(r,\theta,t) = \left[C_n \frac{\gamma L}{a}\right] \frac{2\alpha_{ni}\tanh(\alpha_{ni}\frac{\gamma L}{a})}{(\alpha_{ni}^2 - n^2)} \left[G_n + \frac{1}{2} - \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}} \left(\frac{1}{\tanh(\alpha_{ni}\frac{\gamma L}{a})} - \frac{1}{\sinh(\alpha_{ni}\frac{\gamma L}{a})} + \tanh(\alpha_{ni}\frac{\gamma L}{2a}) \right) \right] \frac{J_n(\alpha_{ni}\frac{r}{a})}{J_n(\alpha_{ni})} \cos n\theta \, g_n(t) \qquad (IV-111)$$

As in the case of the three constant approximation, $\zeta_{ni}(r, \theta, t)$ can be written as

$$\zeta_{ni}(r, \theta, t) = \frac{C_n \gamma L}{a} \bar{\beta}_{ni} J_n \left(\frac{\alpha_{ni} r}{a}\right) \cos n\theta \, q_n(t). \qquad \text{(IV-112)}.$$

$$\bar{\beta}_{ni} = \frac{2\alpha_{ni} \tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)}{(\alpha_{ni}^2 - n^2) J_n(\alpha_{ni})} \left[G_n + \frac{1}{2} - \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}} \left(\frac{1}{\tanh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} - \right.\right.$$

$$\left.\left. - \frac{1}{\sinh\left(\alpha_{ni}\frac{\gamma L}{a}\right)} + \tanh\left(\alpha_{ni}\frac{\gamma L}{2a}\right)\right)\right] \qquad \text{(IV-113)}$$

Table IV-1

Roots α_{ni} for $J_n'(\alpha_{ni}) = 0$

	n = 0	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6
i = 1	3.832	1,841	3.053	4.20	5.31	6.40	7.50
i = 2	7.016	5.332	6.707	7.89	9.04	10.52	11.74
i = 3	10.17	8.536	9.97	11.17	12.33	13.99	15.27
i = 4	13.32	11.71	13.17	14.37	15.53	17.27	18.60
i = 5	16.47	14.86	16.31	17.52	18.79	20.53	21.88

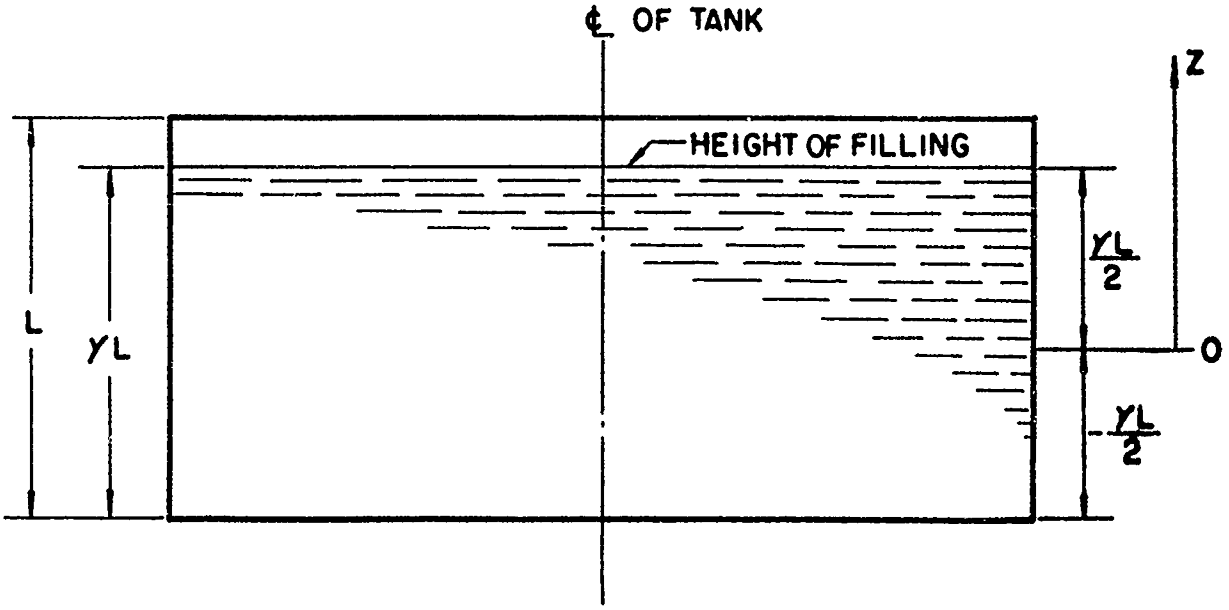

FIG. IV-1 SHELL WITH PARTIAL FILLING

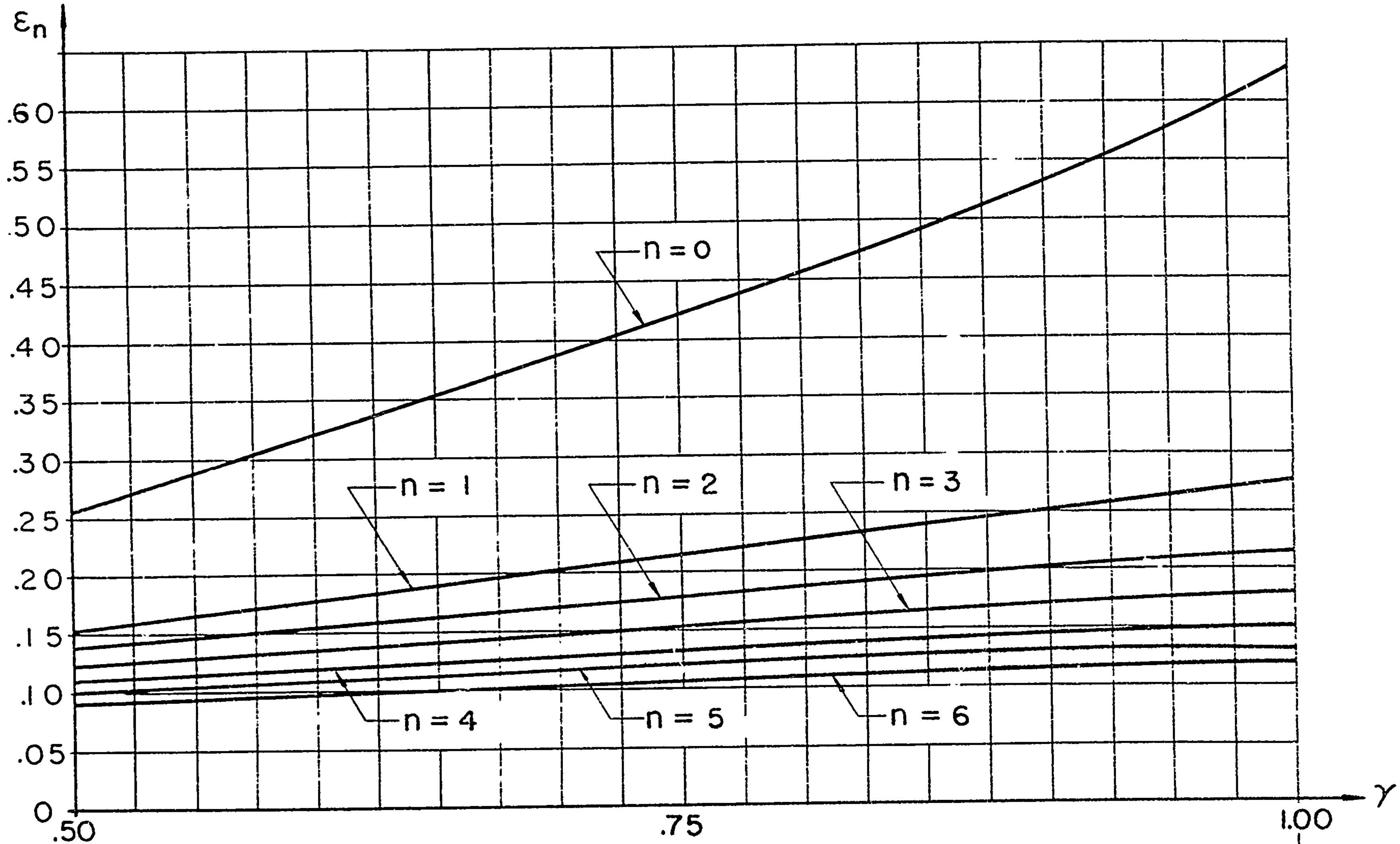

FIG. IV-2 VIRTUAL MASS COEFFICIENT ε_n

FREE SURFACE

THREE CONSTANT APPROXIMATION USING

THREE FLUID TERMS $i = 1, 2, 3$

$L/a = 0.8$ — $m_{vn} = \varepsilon_n \rho a$ — HEIGHT OF FILLING $= \gamma L$

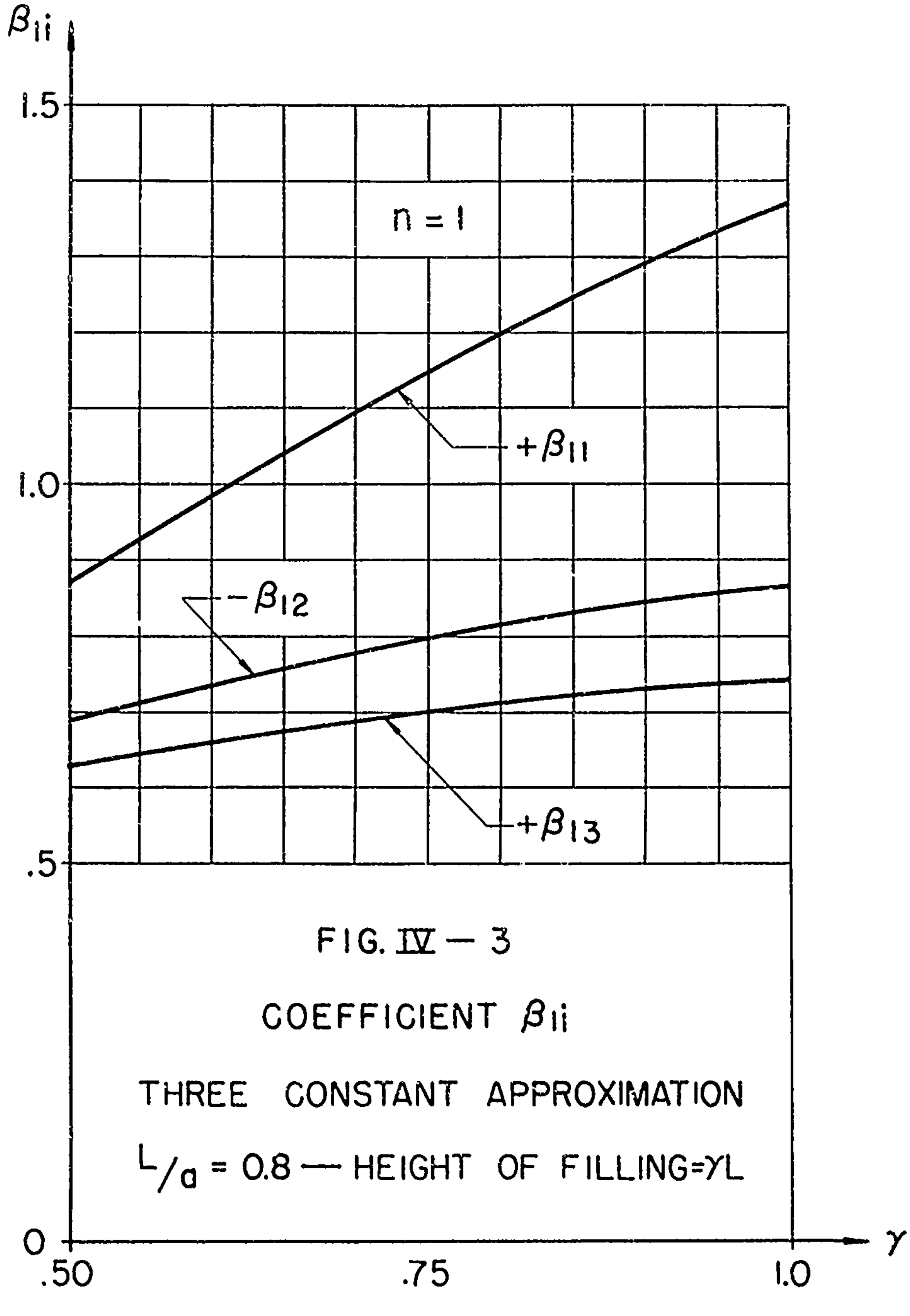

FIG. IV — 3

COEFFICIENT β_{1i}

THREE CONSTANT APPROXIMATION

$L/a = 0.8$ — HEIGHT OF FILLING $= \gamma L$

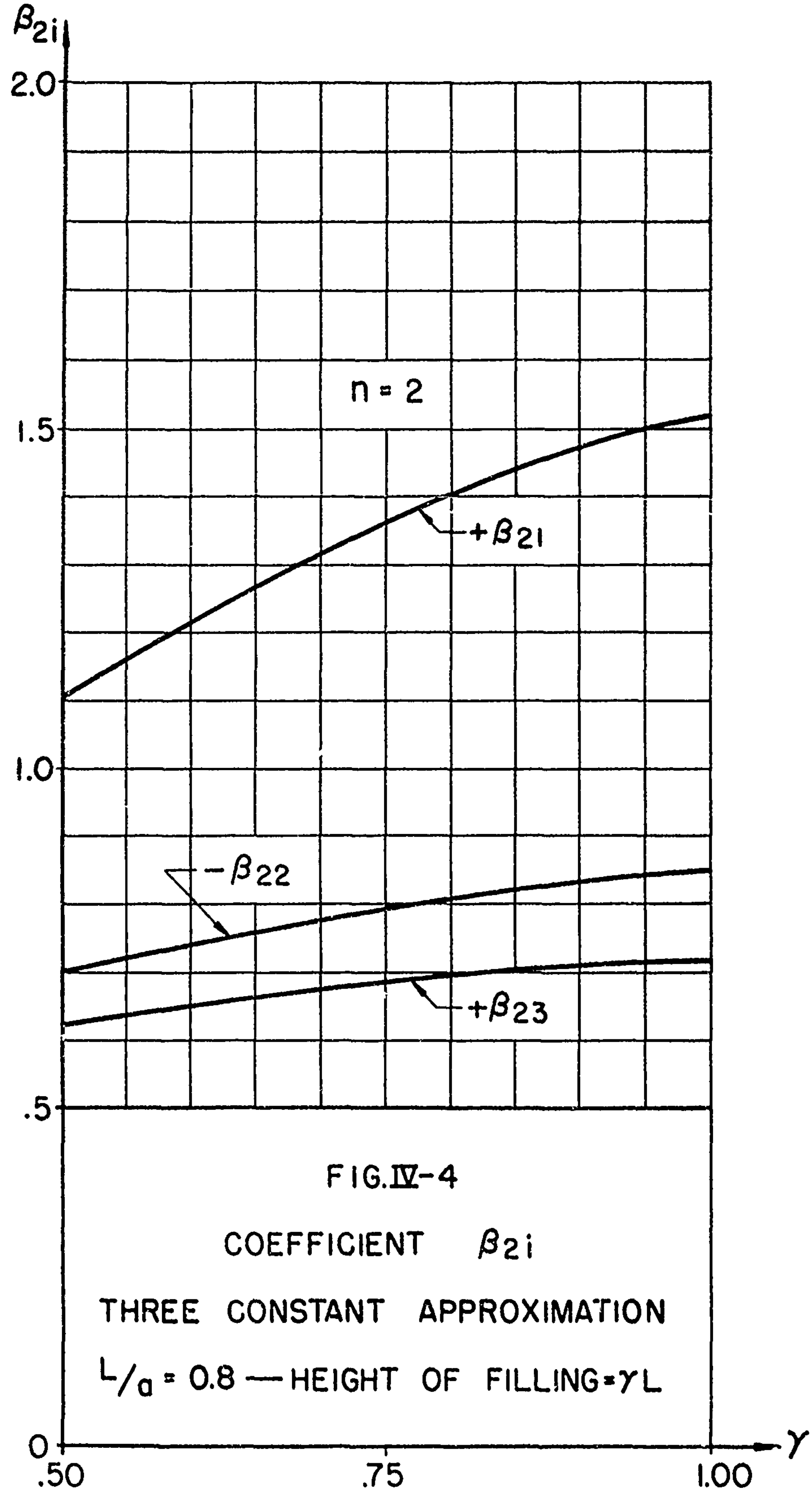

FIG. IV-4

COEFFICIENT β_{2i}

THREE CONSTANT APPROXIMATION

$L/a = 0.8$ — HEIGHT OF FILLING $= \gamma L$

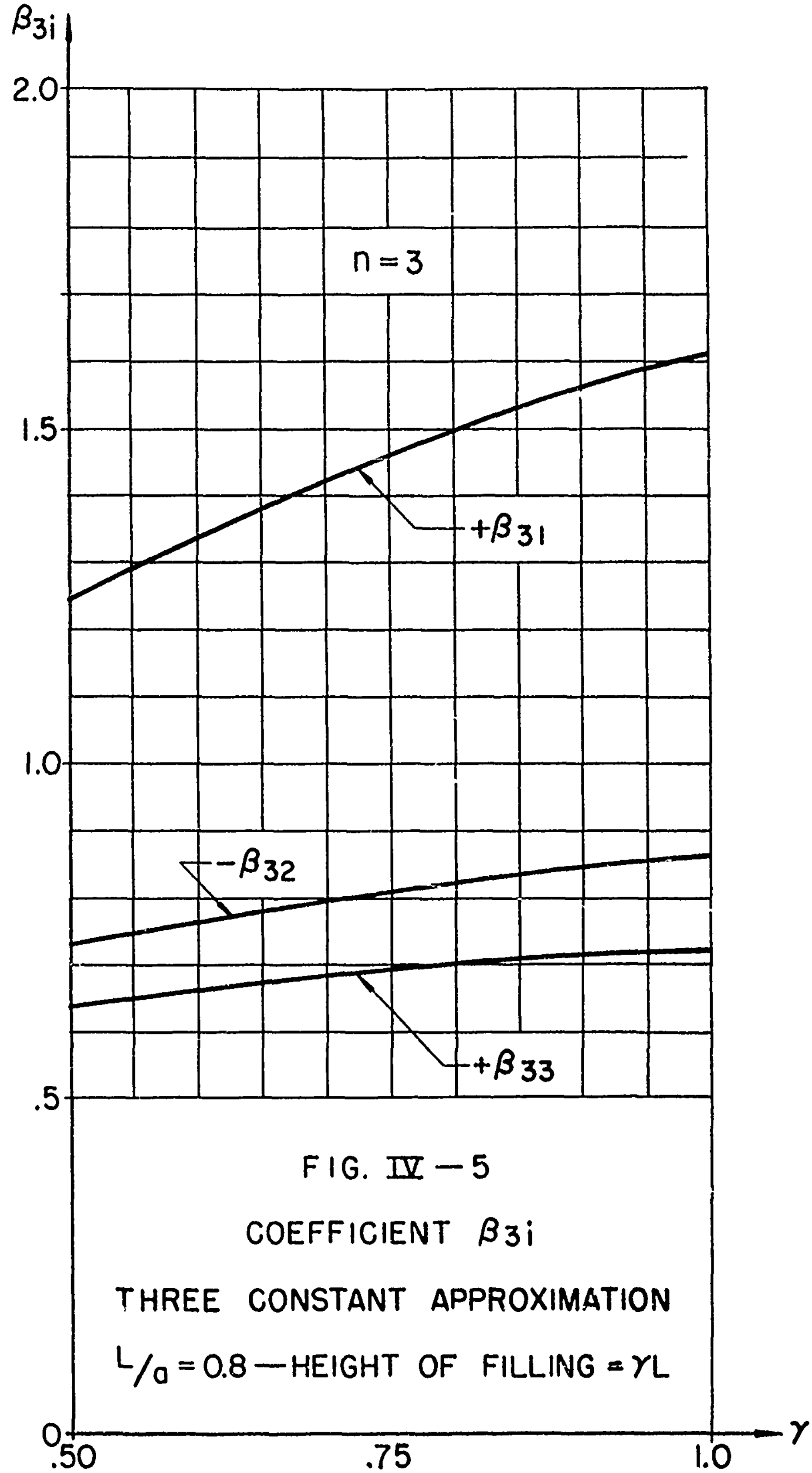

FIG. IV — 5

COEFFICIENT β_{3i}

THREE CONSTANT APPROXIMATION

$L/a = 0.8$ — HEIGHT OF FILLING = γL

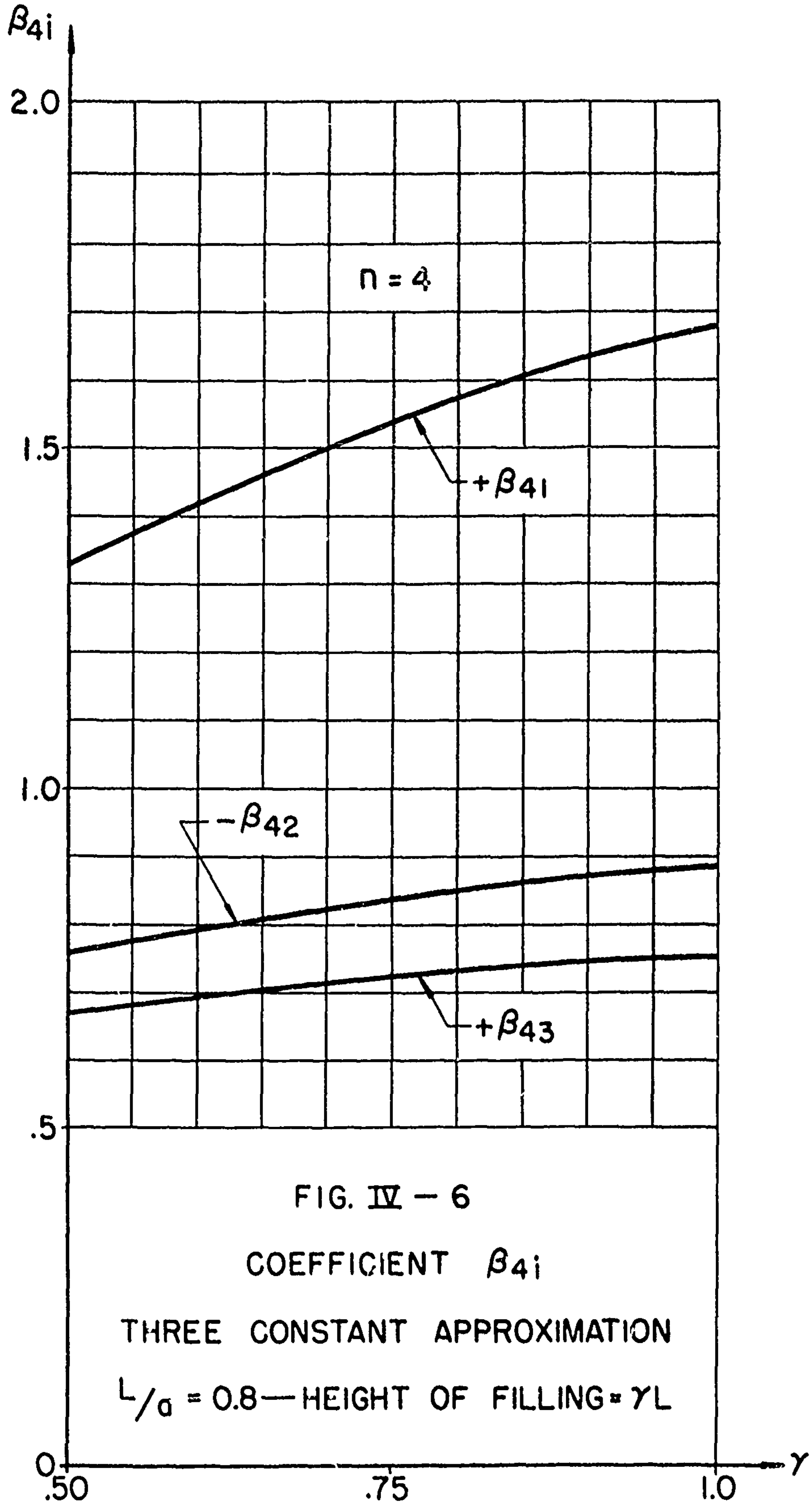

FIG. IV – 6

COEFFICIENT β_{4i}

THREE CONSTANT APPROXIMATION

L/a = 0.8—HEIGHT OF FILLING = γL

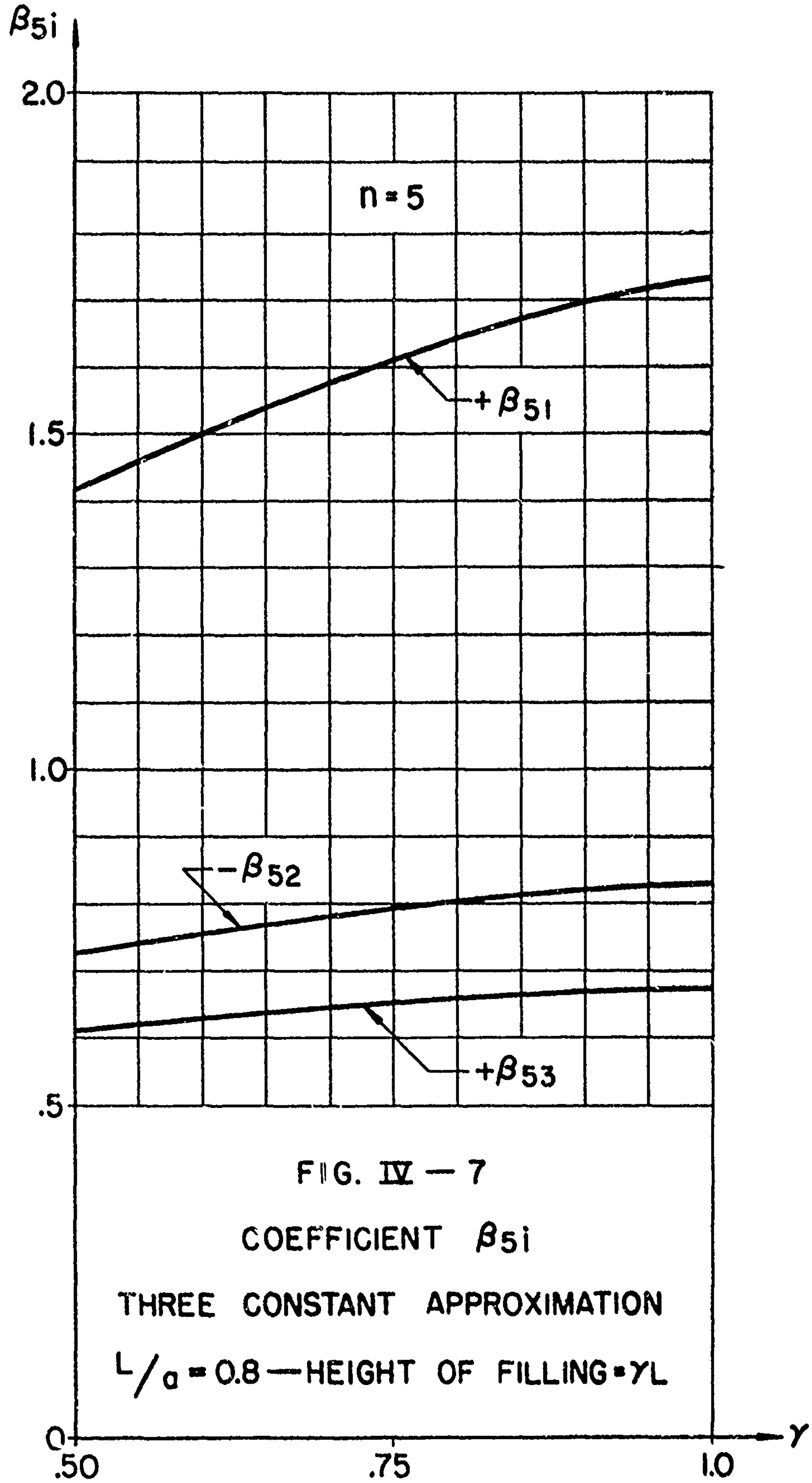

FIG. IV — 7

COEFFICIENT β_{5i}

THREE CONSTANT APPROXIMATION

$L/a = 0.8$ — HEIGHT OF FILLING $= \gamma L$

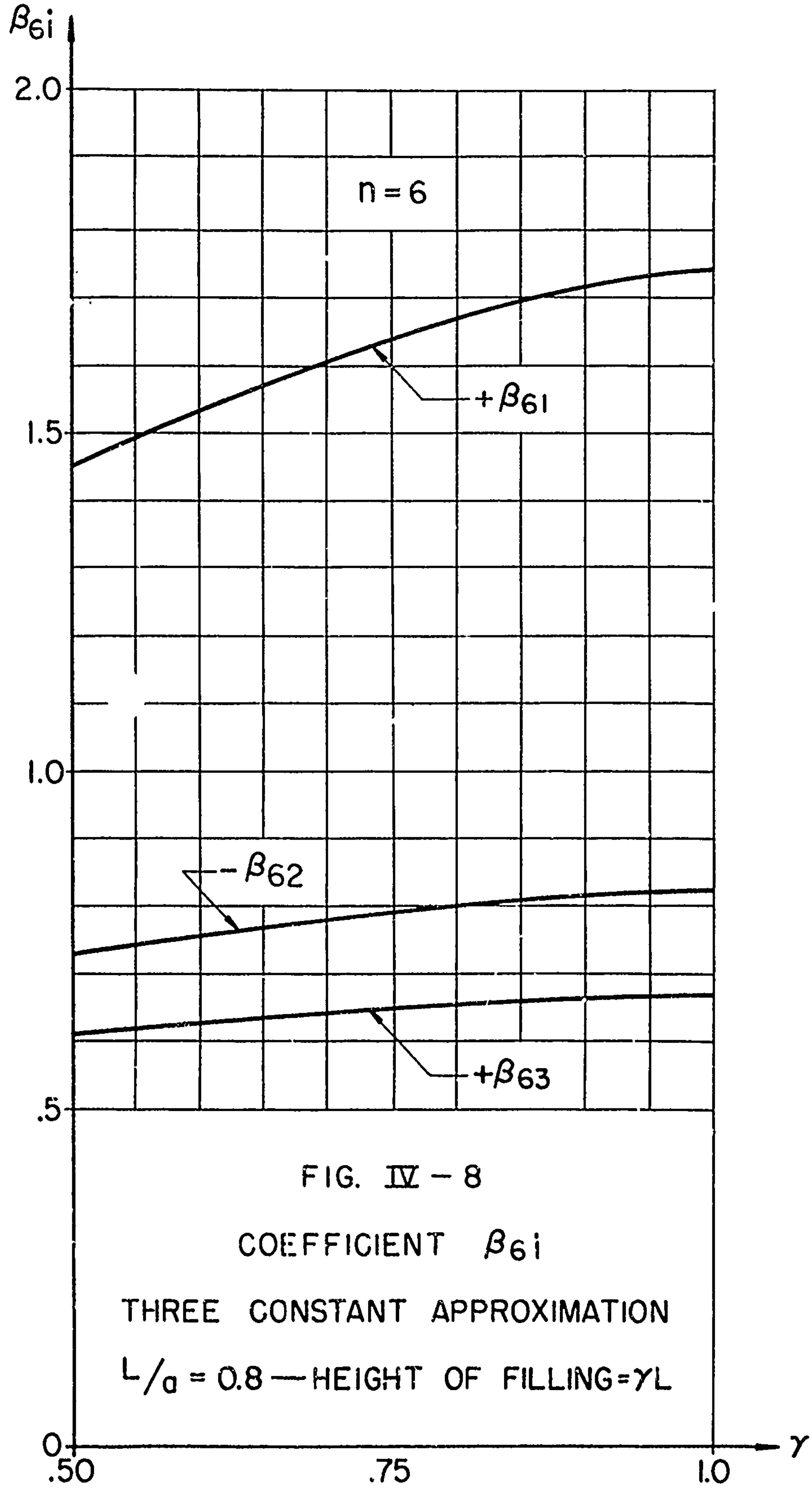

FIG. IV - 8

COEFFICIENT β_{6i}

THREE CONSTANT APPROXIMATION

$L/a = 0.8$ — HEIGHT OF FILLING = γL

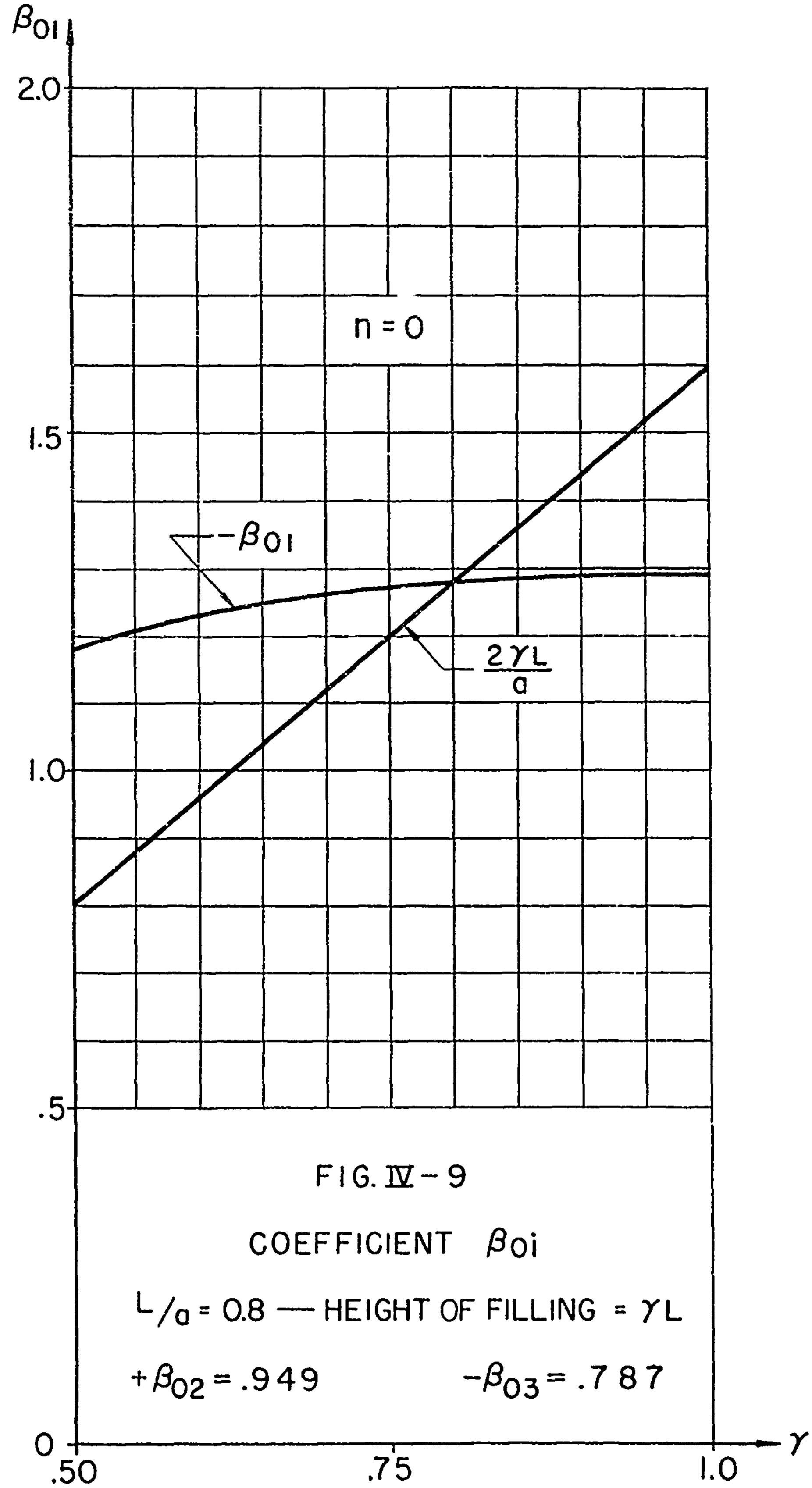

FIG. IV-9

COEFFICIENT β_{0i}

$L/a = 0.8$ — HEIGHT OF FILLING $= \gamma L$

$+\beta_{02} = .949$ $-\beta_{03} = .787$

V. Modes and Frequencies of Free Vibrations of Partially Full Cylindrical Tanks.

Expressions for obtaining the modes and the frequencies of free vibrations of partially full cylindrical tanks are presented in this section of the report. As in the case of an empty tank, two sets of approximate displacements are considered; 1) a set in which five constants are retained in Eq. (I-1) to (I-3) and, 2) a set in which three constants are retained. While the latter will give satisfactory estimates of frequencies for most applications, the former are required to give accurate values of strains and stresses for use in forced vibration problems.

Employing the Rayleigh-Ritz method as in Section I, the total kinetic energy of the partially filled shell in the n^{th} mode is given by the relation

$$T_n = T_{nshell} + T_{nfluid} \tag{V-1}$$

where T_{nshell} is given by Eq. (I-7) and T_{nfluid} is given by Eq. (IV-55) for the three constant approximation or Eq. (IV-108) for the five constant approximation.

The potential energy of the combined system is not effected by the fluid in the shell (the effect of gravity forces on the potential energy of the system has been neglected in Section (IV) and is given by Eq. (I-10)-(I-12).

As in the case of the empty shell, frequency determinants will be given for the case in which both membrane and bending effects are considered,

and for the case of a membrane shell. For the mode $n = 0$, special conditions prevail, and this case will be discussed in part (f) of this section.

a) <u>Approximation Using Five Constants - Membrane and Bending Effects $n \neq 0$</u>

Let the displacements of the shell be given by Eq. (I-4)-(I-6) which are repeated here for convenience.

$$u(z,\ \theta,\ t) = u(z,\ \theta)e^{i\omega t} = \left[U\frac{z}{a} + X\left(\frac{z^2}{a^2} - \frac{3Lz}{4a^2}\right)\right] \cos n\theta\ e^{i\omega t} \tag{V-2}$$

$$v(z,\ \theta,\ t) = v(z,\ \theta)e^{i\omega t} = V\frac{z}{a} \sin n\theta\ \ e^{i\omega t} \tag{V-3}$$

$$w(z,\ \theta,\ t) = w(z,\ \theta)e^{i\omega t} = \left[Y + W\left(\frac{z}{a} - \frac{L}{2a}\right)\right] \cos n\theta\ e^{i\omega t} \tag{V-4}$$

The kinetic energy of the shell mass in the n^{th} mode, T_{nshell}, is given by

$$T_{nshell} = \frac{m_i}{2} \int_0^{2\pi} \int_0^L (\dot{u}_n^2 + \dot{v}_n^2 + \dot{w}_n^2)\ ad\theta dz \tag{V-5}$$

and the kinetic energy of the fluid, T_{nfluid} is given by Equation (IV-108) for the five constant approximation:(1)

$$T_{nfluid} = \frac{m_i \pi a \omega^2}{6} \beta^3 a \left[\eta_n \gamma^3 \frac{\rho a}{m_i} W^2 + \bar{\eta}_n \gamma^3 \frac{\rho a}{m_i} Y^2 + \bar{\bar{\eta}}_n \gamma^3 \frac{\rho a}{m_i} YW\right] \tag{V-6}$$

---

(1) For free vibrations, the time dependency of the displacements is taken as $e^{i\omega t}$.

Substituting Eq. (V-2)-(V-4) into Eq. (V-5) and adding Eq. (V-6), the total kinetic energy of the partially filled shell in the n^{th} mode is obtained from Eq. (V-1):

$$T_n = \frac{m_1 \pi e \omega^2}{6} \xi^3 a \left[U^2 + V^2 + \frac{W^2}{4} \left(1 + 4\eta_n \gamma^3 \frac{\rho a}{m_1}\right) + \frac{3}{80} \xi^2 X^2 + \frac{3}{\xi^2} \left(1 + \frac{\bar{\eta}_n \gamma^3 \xi^2}{3} \frac{\rho a}{m_1}\right) Y^2 + \bar{\bar{\eta}}_n \gamma^3 \frac{\rho a}{m_1} YW \right] \tag{V-7}$$

where the coefficients η_n , $\bar{\eta}_n$ and $\bar{\bar{\eta}}_n$ are given by Eqs. (IV-109a)-(IV-109c) respectively.

The potential energy V_n stored in the shell can be expressed as a function of the displacements $u(z, \theta)$, $v(z, \theta)$ and $w(z, \theta)$ by Eq. (I-10)-(I-12), which upon substitution of Eq. (V-2)-(V-4) become the expressions given by Eq. (I-13) and Eq. (I-14).

Applying the Rayleigh-Ritz method, the following set of five homogeneous linear equations in the five unknowns U, V, W, X and Y are obtained:

$$\left[2M - \frac{6}{\xi^2} - (1-\nu)n^2 - k(1-\nu)n^2\right] U + \left[-\frac{3n(3\nu-1)}{2\xi}\right] V + \left[\frac{k3(1-\nu)n^2}{2\xi}\right] W + \left[-\frac{3}{2\xi}\right] X + \left[\frac{6\nu}{\xi^2}\right] Y = 0 \tag{V-8}$$

$$\left[-\frac{3n(3\nu-1)}{2\xi}\right] U + \left[2M - \frac{3(1-\nu)}{\xi^2} - 2n^2 - \frac{k9(1-\nu)}{\xi^2}\right] V + \left[\frac{n}{2} \frac{k9n(1-\nu)}{\xi^2}\right] W + \left[-\frac{n}{8}(1+13\nu)\right] X + \left[\frac{3n}{\xi}\right] Y = 0 \tag{V-9}$$

$$\left[\frac{k3(1-\nu)n^2}{2\xi}\right] U + \left[\frac{n}{2} + \frac{k9n(1-\nu)}{\xi^2}\right] V + \left[\frac{M}{2}\left\{1 + 4\eta_n\gamma^3\frac{\rho a}{m_1}\right\} - \frac{1}{2} - \right.$$

$$\left. - k\left\{\frac{(1-n^2)^2}{2} + \frac{12(1-\nu)n^2}{\xi^2}\right\}\right] W + \left[\nu - k\left(\frac{1-\nu}{8}\right)n^2\right] X + \left[\gamma^3\bar{\bar{\eta}}_n\frac{\rho a}{m_1} M\right] Y = 0$$

(V-10)

$$-\left[\frac{3}{2\xi}\right] U + \left[-\frac{n(1+13\nu)}{8}\right] V + \left[\nu - k\left(\frac{1-\nu}{8}\right)n^2\right] W +$$

$$+\left[\frac{3}{40}\xi^2 M - \frac{19}{8} - \frac{3(1-\nu)n^2\xi^2}{80} - \frac{k3(1-\nu)n^2\xi^2}{80}\right] X + \left[\frac{3}{2\xi}\right] Y = 0$$

(V-11)

$$\left[\frac{\nu 6}{\xi^2}\right] U + \left[\frac{3n}{\xi}\right] V + \left[\gamma^3\bar{\bar{\eta}}_n\frac{\rho a}{m_1} M\right] W + \left[\frac{3\nu}{2\xi}\right] X$$

$$+\left[\frac{6}{\xi^2}\left\{M\left(1 + \frac{\gamma^3\xi^2}{3}\frac{\rho a}{m_1}\bar{\eta}_n\right) - 1 - k(1-n^2)^2\right\}\right] Y = 0$$

(V-12)

where k and M are defined by Eq. (I-21) and (I-22) respectively, γ is the fractional height of the filling $\xi = \frac{L}{a}$ and η_n, $\bar{\eta}_n$, and $\bar{\bar{\eta}}_n$ are defined by Eq. (IV-109a)-(IV-109c).

Nonvanishing solutions of Eq. (V-8)-(V-12) and free vibrations exist only if the determinant of the system vanishes. This leads to the determinantal frequency equation shown on Page 96.

For given values of ξ, n and γ, Eq. (V-13) yields five positive roots M_j, defining five mutually orthogonal modes, of frequencies

$$\omega_j^2 = \frac{M_j E h}{m_1 a^2(1-\nu^2)} \qquad j = 1, 2, 3, 4, 5. \qquad \text{(V-14)}$$

$$
\begin{vmatrix}
2M-\frac{6}{\xi^2}-(1-\nu)n^2(1+k) & -\frac{3n(3\nu-1)}{2\xi} & \frac{3kn^2(1-\nu)}{2\xi} & -\frac{3}{2\xi} & \frac{6\nu}{\xi^2} \\
-\frac{3n(3\nu-1)}{2\xi} & 2M-2n^2-\frac{3(1-\nu)}{\xi^2}(1+3k) & \frac{n}{2}+\frac{knq(1-\nu)}{\xi^2} & -\frac{n(1+13\nu)}{8} & \frac{3n}{\xi} \\
\frac{3kn^2(1-\nu)}{2\xi} & \frac{n}{2}+\frac{knq(1-\nu)}{\xi^2} & \frac{M}{2}\left\{1+4\eta_n\gamma^3\frac{\rho a}{m_i}\right\}-\frac{1}{2}-k\left\{\frac{(1-n^2)^2}{2}+\frac{12(1-\nu)n^2}{\xi^2}\right\} & \nu-\frac{k(1-\nu)n^2}{8} & \frac{\gamma^3\bar{\bar{\eta}}_n}{\xi}\frac{\rho a}{m_i}M \\
-\frac{3}{2\xi} & -\frac{n(1+13\nu)}{8} & \nu-\frac{k(1-\nu)n^2}{8} & \frac{3}{40}M\xi^2-\frac{19}{8}-\frac{3(1-\nu)n^2\xi^2}{80}(1+k) & \frac{3\nu}{2\xi} \\
\frac{6\nu}{\xi^2} & \frac{3n}{\xi} & \frac{\gamma^3\bar{\bar{\eta}}_n}{\xi}\frac{\rho a}{m_i}M & \frac{3\nu}{2\xi} & \frac{6}{\xi^2}\left\{M\left(1+\frac{\gamma^3\xi^2\rho a\bar{\bar{\eta}}_n}{3m_i}\right)-1-k(1-n^2)^2\right\}
\end{vmatrix} = 0
$$

$n \neq 0$

Membrane + Bending Effects
5 Constant Approximation
Partially Filled Shell

γL = height of filling

(V-13)

The shape of the mode pertaining to a particular frequency ω_j can be found by computing the ratios $\frac{U}{W}$, $\frac{V}{W}$, $\frac{X}{W}$ and $\frac{Y}{W}$ from any four of the Equations (V-8) to (V-12). As in the case of the empty tank, in general, only the lowest frequency ω_1 will be required and for simplicity, the subscript j will be dropped for each of the constants U_j, V_j, W_j, X_j and Y_j.

Once the frequency ω and the ratios of the constants have been evaluated for a particular value of n, the displacements of the shell become:

$$u(z,\ \theta) = C_n \left[\frac{U}{W}\ \frac{z}{a} + \frac{X}{W}\left(\frac{z^2}{a^2} - \frac{3Lz}{4a^2}\right) \right] \cos n\theta \qquad \text{(V-15)}$$

$$v(z,\ \theta) = C_n \left[\frac{V}{W}\ \frac{z}{a} \right] \sin n\theta \qquad \text{(V-16)}$$

$$w(z,\ \theta) = C_n \left[\frac{Y}{W} + \left(\frac{z}{a} - \frac{L}{2a}\right) \right] \cos n\theta \qquad \text{(V-17)}$$

The constant C_n is again chosen so as to normalize the mode to the total mass of the empty shell. The normalization condition thus becomes:

$$m_1 \int_0^L \int_0^{2\pi} (u_n^2 + v_n^2 + w_n^2)\ a\,d\theta\,dz + 2\ \overline{T}_{fluid} = 2\pi a L m_1 = M_1 \qquad \text{(V-18)}$$

where $\overline{T}_{fluid}$ is the space dependent part of Eq. (IV-108). Substituting Eq. (V-15)-(V-17) into Eq. (V-18), the expression for the normalization coefficient C_n is

$$C_n = \left\{ \frac{\xi^2}{6} \left[\left(\frac{U}{W}\right)^2 + \left(\frac{V}{W}\right)^2 + \frac{1}{4} + \eta_n \gamma^3 \frac{\rho a}{m_1} + \frac{3}{80}\xi^2 \left(\frac{X}{W}\right)^2 + \right.\right.$$
$$\left.\left. + \frac{3}{\xi^2}\left(1 + \frac{\bar{\eta}_n \gamma^3 \xi^2 \rho a}{3m_1}\right)\left(\frac{Y}{W}\right)^2 + \bar{\bar{\eta}}_n \gamma^3 \frac{\rho a}{m_1}\left(\frac{Y}{W}\right) \right] \right\}^{-1/2} \qquad \text{(V-19)}$$

Using Eq. (V-19), Eq. (V-15)-(V-17) give the normalized modes of free vibrations of the partially filled tank.

The normalized strains and stresses in the partially filled tank may be evaluated from Eq. (I-30)-(I-35), using the ratios of the constants and the expression for the normalization coefficient that have been derived in the present section.

b. Approximation Using Five Constants-Membrane Effects Only.

$n \neq 0$

As in the case of empty shells, a membrane approximation may be made in which the strain energy of bending, V_2, is set equal to zero. For thin shells in which the thickness to radius ratio, h/a, is very small, excellent approximations of frequencies, strains and stresses can be obtained from such a procedure for modes with a low circumferential wave number "n". It may be noted that the range of h/a and n for which the membrane assumption is permissible can be estimated by methods similar to those of Section III of this Report.

The frequency equation and the mode shapes for the partially full membrane shell are obtained by setting the value of the constant k equal to zero in Eq. (V-7)-(V-11). The system of homogeneous equations then becomes:

$$\left[2M - \frac{6}{\xi^2} - (1-\nu)n^2\right] U + \left[-\frac{3n(3\nu-1)}{2\xi}\right] V + \left[0\right] W + \left[-\frac{3}{2\xi}\right] X + \left[\frac{6\nu}{\xi^2}\right] Y = 0 \qquad \text{(V-20)}$$

$$\left[-\frac{3n(3\nu-1)}{2\xi}\right] U + \left[2M - \frac{3(1-\nu)}{\xi^2} - 2n^2\right] V + \left[\frac{n}{2}\right] W + \left[-\frac{n}{8}(1+13\nu)\right] X + \left[\frac{3n}{\xi}\right] Y = 0 \qquad \text{(V-21)}$$

$$\left[0\right] U + \left[\frac{n}{2}\right] V + \left[\frac{M}{2}\left\{1 + 4\eta_n\gamma^3\frac{\rho a}{m_1}\right\} - \frac{1}{2}\right] W + \left[\nu\right] X + \left[\gamma^3\bar{\eta}_n\frac{\rho a}{m_1} M\right] Y = 0 \qquad \text{(V-22)}$$

$$\left[-\frac{3}{2\xi}\right] U + \left[-\frac{n(1+13\nu)}{8}\right] V + \left[\nu\right] W + \left[\frac{3}{40}\xi^2 M - \frac{19}{8} - \frac{3(1-\nu)n^2\xi^2}{80}\right] X + \left[\frac{3\nu}{2\xi}\right] Y = 0 \qquad \text{(V-23)}$$

$$\left[\frac{\nu 6}{\xi^2}\right] U + \left[\frac{3n}{\xi}\right] V + \left[\gamma^3 \bar{\eta}_n \frac{\rho a}{m_i} M\right] W + \left[\frac{\nu 3}{2\xi}\right] X + \left[\frac{6}{\xi^2} \left\{ M\left(1 + \frac{\gamma^3 \xi^2}{3} \frac{\rho a}{m_i} \bar{\eta}_n\right) - 1\right\}\right] Y = 0 \qquad \text{(V-24)}$$

The determinantal frequency equation is obtained by setting the determinant of the above system equal to zero, or by setting $k = 0$ in Eq. (V-13). For this determinant see Page 101. For given values of ξ, n and γ, Eq. (V-25) yields the five positive roots M_j. The shape of the mode pertaining to a particular M_j can be found by computing the ratios $\frac{U}{W}$, $\frac{V}{W}$, $\frac{X}{W}$ and $\frac{Y}{W}$ from any four of the Eq. (V-20)-(V-24).

Again only the frequency corresponding to the lowest root M_1, will be of interest.

Once the frequency ω and the ratios of the constants have been evaluated for a particular value of n, Eq. (V-14)-(V-18) may be used to compute the normalized mode shapes of the tank. The normalized strains and stresses in the partially filled tank may then be evaluated from Eq. (I-30)-(I-35), using the normalization constant C_n and the mode shapes of the present section.

$$
\begin{vmatrix}
2M-\frac{6}{\xi^2}-(1-\nu)n^2 & -\frac{3n(3\nu-1)}{2\xi} & 0 & -\frac{3}{2\xi} & \frac{6\nu}{\xi^2} \\
-\frac{3n(3\nu-1)}{2\xi} & 2M-2n^2-\frac{3(1-\nu)}{\xi^2} & \frac{n}{2} & -\frac{n(1+13\nu)}{8} & \frac{3n}{\xi} \\
0 & \frac{n}{2} & \frac{M}{2}\left\{1+4\bar{\eta}_n\gamma^3\frac{\rho a}{m_i}\right\}-\frac{1}{2} & \nu & \gamma^3\bar{\bar{\eta}}_n\frac{\rho a}{m_i}M \\
-\frac{3}{2\xi} & -\frac{n(1+13\nu)}{8} & \nu & \frac{3}{40}M\xi^2-\frac{19}{8}-\frac{3(1-\nu)n^2\xi^2}{80} & \frac{3\nu}{2\xi} \\
\frac{6\nu}{\xi^2} & \frac{3n}{\xi} & \gamma^3\bar{\bar{\eta}}_n\frac{\rho a}{m_i}M & \frac{3\nu}{2\xi} & \frac{6}{\xi^2}\left\{M\left(1+\frac{\gamma^3\xi^2}{3}\frac{\rho a}{m_i}\bar{\eta}_{(n)}\right)-1\right\}
\end{vmatrix} =
$$

$n \neq 0$

Membrane Effects
5 Constant Approximation
Partially Filled Shell
γL = height of filling

(V-25)

c) Approximation Using Three Constants - Membrane and Bending Effects.

$n \neq 0$

Let the displacements of the shell be given by Eq. (I-42)-(I-44) which are repeated here for convenience:

$$u(z, \theta, t) = u(z, \theta)\, e^{i\omega t} = U \frac{z}{a} \cos n\theta\, e^{i\omega t} \qquad \text{(V-26)}$$

$$v(z, \theta, t) = v(z, \theta)\, e^{i\omega t} = V \frac{z}{a} \sin n\theta\, e^{i\omega t} \qquad \text{(V-27)}$$

$$w(z, \theta, t) = w(z, \theta)\, e^{i\omega t} = W \frac{z}{a} \cos n\theta\, e^{i\omega t} \qquad \text{(V-28)}$$

The kinetic energy of the partially filled shell in the n^{th} mode is given by Eq. (IV-56) where m_{vn} is given by Eq. (IV-57). Substituting Eq. (V-26)-(V-28) with Eq. (V-5) the kinetic energy becomes

$$T_n = \frac{m_1 \pi \omega^2 a}{6} \frac{L^3}{a^2} \left[U^2 + V^2 + W^2(1 + K_n \gamma^3) \right] \qquad \text{(V-29)}$$

where

$$K_n = \frac{m_{vn}}{m_1} = \epsilon_n \frac{\rho a}{m_1} \quad . \qquad \text{(V-29a)}$$

The potential energy V_n can be expressed as a function of the displacements by substituting Eq. (V-26)-(V-28) into Eq. (I-10)-(I-12), thus obtaining the expressions given by Eq. (I-46) and (I-47).

Applying the Rayleigh-Ritz method, the following set of three homogeneous linear equations in the unknowns, U, V, and W is obtained.

$$\left[2M - \frac{6}{\xi^2} - n^2(1-\nu)(1+\kappa)\right] U + \left[\frac{-3n(3\nu-1)}{2\xi}\right] V + \left[\frac{3}{\xi}\left\{\nu + \frac{kn^2(1-\nu)}{2}\right\}\right] W = 0$$

(V-30)

$$\left[\frac{-3n(3\nu-1)}{2\xi}\right] U + \left[2M - 2n^2 - \frac{3}{\xi^2}(1-\nu)(1+3k)\right] V + \left[2n + \frac{9kn(1-\nu)}{\xi^2}\right] W = 0$$

(V-31)

$$\left[\frac{3}{\xi}\left\{\nu + \frac{kn^2(1-\nu)}{2}\right\}\right] U + \left[2n + \frac{9kn(1-\nu)}{\xi^2}\right] V$$

$$+ \left[2M(1+K_n\gamma^3) - 2 - \frac{6k}{\xi^2}\left\{(1-n^2)^2 \frac{\xi^2}{3} + 2(1-\nu)n^2\right\}\right] W = 0 \qquad \text{(V-32)}$$

where k and M are defined by Eq. (I-21) and (I-22), and γ is the fractional height of the filling.

Nonvanishing solutions of Eq. (V-30)-(V-32) exist only if the determinant of the system vanishes, thus leading to the determinantal frequency equation shown on Page 104.

For given values of ξ, n and γ, Eq. (V-33) yields the three positive roots M_j, defining three mutually orthogonal modes, of frequencies given by Eq. (V-14). The shape of the mode pertaining to a particular frequency ω_j can be obtained by computing the ratios $\frac{U}{W}$ and $\frac{V}{W}$ from any two of Eq. (V-30)-(V-32).

As in the case of the empty tank, the lowest frequency ω_1 only will be required and the subscript j will be dropped for simplicity.

The displacements of the shell may be written in the form

$$u(z, \theta) = C_n \frac{U}{W} \frac{z}{a} \cos n\theta \qquad \text{(V-34)}$$

$$v(z, \theta) = C_n \frac{V}{W} \frac{z}{a} \sin n\theta \qquad \text{(V-35)}$$

$$w(z, \theta) = C_n \frac{z}{a} \cos n\theta \qquad \text{(V-36)}$$

where the constant C_n is again chosen so as to normalize the mode to the total mass of the empty shell. The normalization condition is given by Eq. (V-18),[2] which upon substitution of Eq. (V-34)-(V-36) gives the following expression for the normalization coefficient C_n:

(2) For the three constant approximation, $\bar{T}_{fluid}$ is the space dependent part of Eq. (IV-55).

$$
\begin{vmatrix}
2M - \dfrac{6}{\xi^2} - n^2(1-\nu)(1+k) & -\dfrac{3n(3\nu-1)}{2\xi} & \dfrac{3}{\xi}\left\{\nu + \dfrac{kn^2(1-\nu)}{2}\right\} \\
-\dfrac{3n(3\nu-1)}{2\xi} & 2M - 2n^2 - \dfrac{3(1-\nu)(1+3k)}{\xi^2} & 2n + \dfrac{9kn(1-\nu)}{\xi^2} \\
\dfrac{3}{\xi}\left\{\nu + \dfrac{kn^2(1-\nu)}{2}\right\} & 2n + \dfrac{9kn(1-\nu)}{\xi^2} & 2M(1+K_n\gamma^3) - 2 - \dfrac{6k}{\xi^2}\left\{(1-n^2)^2\dfrac{\xi^2}{3} + 2(1-\nu)n^2\right\}
\end{vmatrix} = 0
$$

$n \neq 0$

Membrane + Bending Effects
3 Constant Approximation
Partially Filled Tank

γL = height of filling

(V-33)

$$C_n = \left\{ \frac{6}{\xi^2} \left[\frac{1}{1 + K_n \gamma^3 + (\frac{U}{W})^2 + (\frac{V}{W})^2} \right] \right\}^{1/2} \qquad \text{(V-37)}$$

Using Eq. (V-37), Eq. (V-34)-(V-36) give the normalized modes of free vibration of the partially filled tank.

The results obtained from this approximation may be used where an estimate of the frequency of vibration of any particular mode is required. However, the present approximation does not give sufficiently accurate results for the strains and stresses in the shell, and the approximation using five constants must be employed when these quantities are required.

d) Approximation Using Three Constants - Membrane Effects Only.

$n \neq 0$

As in Part (b) of this section a membrane approximation may be used to obtain the frequencies of sufficiently thin shells in the lower modes of n. The frequency equation and the mode shapes for the membrane shell are obtained by setting the value of the constant k equal to zero in Eq. (V-30)-(V-33). The determinantal frequency is given on Page 107.

The remarks of Part (c) of this section regarding the validity and applicability of the three constant approximation also hold for this case.

$$\begin{vmatrix} 2M - \frac{6}{\xi^2} - n^2(1-\nu) & -\frac{3n(3\nu-1)}{2\xi} & \frac{3\nu}{\xi} \\ -\frac{3n(3\nu-1)}{2\xi} & 2M - 2n^2 - \frac{3(1-\nu)}{\xi^2} & 2n \\ \frac{3\nu}{\xi} & 2n & 2M(1+K_n\gamma^3)-2 \end{vmatrix} = 0$$

$n \neq 0$

Membrane Effects
3 Constant Approximation
Partially Filled Tank

γL = height of filling

(V-33a)

e) <u>Determination of Approximate Frequencies of Partially Full Cylindrical Tanks - Rayleigh's Method.</u>

If the frequencies and mode shapes of an empty cylindrical tank are known, an expression for the determination of approximate values of the frequencies of the partially full tank, in terms of these quantities can be derived from Rayleigh's principle. For a partially full tank, Rayleigh's principle may be written for the n^{th} mode of vibration:

$$T_{n\,max} = T_{nshell,\ max} + T_{nfluid,\ max} = V_{n,\ max} \qquad \text{(V-38)}$$

where the subscript "max" indicates the maximum value of the respective quantity and the T_{nshell}, T_{nfluid} and V_n are given by Eq. (I-7), (IV-55) and (I-10) respectively. Writing $T_{n,\ max}$ in the form

$$T_{n,\ max} = \omega_n^2\, \overline{T}_{n,\ max} \qquad \text{(V-39)}$$

where $\overline{T}_n$ is obtained by substituting the displacements $u(z, \theta)$, $v(z, \theta)$ and $w(z, \theta)$ instead of the velocities, into the equations for kinetic energy. Substituting Eq. (V-39) into Eq. (V-38), the frequency of the n^{th} mode of vibration of the partially full shell can be written in the following form:

$$\omega_n^2 = \frac{V_{n,max}}{\overline{T}_{nshell,\ max}} \left[\frac{\overline{T}_{nshell,\ max}}{\overline{T}_{nshell,\ max} + \overline{T}_{nfluid,\ max}} \right] . \qquad \text{(V-40)}$$

Substituting the mode shapes for the empty shell into the expressions for $\overline{T}_n$ and V_n in Eq. (V-40), and noting that for this case, the frequency of the n^{th} mode of the empty tank, ω_{ne}, is given by

$$\omega_{ne}^2 = \frac{V_{n,max}}{\overline{T}_{nshell,max}} , \qquad \text{(V-41)}$$

the frequency equation, Eq. (V-40) becomes

$$\omega_n^2 = \delta_n \omega_{ne}^2 \tag{V-42}$$

where

$$\delta_n = \frac{\overline{T}_{nshell,\ max}}{\overline{T}_{nshell,\ max} + \overline{T}_{nfluid,\ max}} \tag{V-43}$$

The value of δ_n for the five constant approximation is given by the relation,

$$\delta_n = \frac{1}{1 + \frac{C_n^2 \xi^2 \gamma^3}{6}\left[\eta_n \frac{\rho a}{m_1} + \overline{\eta}_n \frac{\rho a}{m_1}\left(\frac{Y}{W}\right)^2 + \overline{\overline{\eta}}_n \frac{\rho a}{m_1}\frac{Y}{W}\right]} \tag{V-44}$$

where C_n and η_n, $\overline{\eta}_n$, $\overline{\overline{\eta}}_n$ are given by Eq. (I-29) and Eq. (IV-109a)-(109c) respectively.

The value of δ_n for the three constant approximation is given by the relation

$$\delta_n = \frac{1}{1 + \frac{C_n^2 K_n \xi^2 \gamma^3}{6}} \tag{V-45}$$

where C_n and K_n are given by Eq. (I-56) and Eq. (V-29a) respectively.

Equations (V-42)-(V-45) may be used to obtain approximate values of the frequency ω_n of the partially filled tank for modes in which $n \geq 2$. It should be emphasized however that for the mode $n = 1$ and for all modes in which the strains and stresses in the shell are requir ed, the value of the frequency should be computed from the determinantal frequency equations given in Sections (V-a)-(V-d) and the correct mode shapes should be evaluated from the corresponding equations given in these sections.

Table (V-1) shows the application of Equations (V-41)-(V-45) to the cases of 9/10 full steel and concrete tanks. The value of the frequency number M_n, which is proportional to ω_n^2 is evaluated from the relation

$$M_n = \delta_n M_{ne} \tag{V-46}$$

where M_{ne} is the frequency number for the empty shell. It is seen that good approximations to the frequency are obtained in both cases.

f) n = 0

The frequency ω_o of a partially full tank can be determined in terms of the frequency ω_{oe} of the empty tank. Noting that the displacement of the shell in the mode n = 0 is purely radial, the total effect of the fluid filling is an increase in the mass of the shell, m_1.

Considering the z-independent radial motion of the shell as described in Section (IV-b),

$$w = q_o(t) \tag{V-47}$$

and using Eq. (IV-88) and (IV-89), the increase in the mass of the shell due to the fluid filling of height γL is given by γm_{vo}. The frequency of the partially full fuel tank can then be computed from the relation

$$\omega_o^2 = \omega_{oe}^2 \left[\frac{1}{1 + \frac{\gamma m_{vo}}{m_1}} \right] \tag{V-48}$$

where

$$\omega_{oe}^2 = \frac{Eh}{(1-\nu^2) m_1 a^2} \tag{V-49}$$

Table V-1

Determination of $M_n = \delta_n M_{ne}$

Unprotected Steel Tank 9/10 Full

$L/a = 0.8, \frac{h}{a} = \frac{1}{1200}, \gamma = .9$

n	M_{ne} Eq.(I-41)	δ_n Eq. (V-44)	M_n Eq.(V-46)	M_n Eq. (V-25)
2	.1906	.0495	.00943	.00809
3	.0859	.0578	.00497	.00475
4	.0432	.0659	.00285	.00281
5	.0241	.0741	.00179	.00175
6	.0147	.0816	.00120	.00120

Protected Steel Tank 9/10 Full

$L/a = 0.8, \frac{h}{a} = \frac{1}{31}, \gamma = .9$

n	M_{ne} Eq.(I-23)	δ_n Eq.(V-44)	M_n Eq.(V-46)	M_n Eq.(V-13)
2	.1911	.3859	.0737	.0688
3	.0934	.4229	.0395	.0387
4	.0659	.4560	.0301	.0301

VI. Modes and Frequencies of Cylindrical Tanks with Windgirders (1)

In Section (I)-Section (V) of this report, approximate expressions for the frequencies and modes of empty and partially filled cylindrical storage tanks were developed. In obtaining these expressions, the effect of the wind stiffening ring, i. e. windgirder, which is generally placed at or near the top of the structure was neglected. The effect of the windgirder on the frequencies obtained in the previous Sections can be derived from a procedure using Rayleigh's Method. An upper bound to the frequency of the tank with a windgirder in terms of the frequency ω_{ns} of the storage tank without a windgirder is obtained. It will be shown that for the type of tanks under consideration, about 40 feet high-100 feet diameter, the effect of the windgirder on the tank frequency is very small. In general, it may be neglected in computations and the formulas presented in the previous Sections of this report can thus be used for the determination of frequencies and modes of free vibrations of structures with windgirders.(2)

(1) The dimensions and the type of windgirders in the analysis are taken from the American Petroleum Institute Specification for Welded Oil Storage Tanks, American Petroleum Institute Standard 12C, Fifteenth Edition, March 1958, American Petroleum Institute, New York, Pg. 13 and 70.

(2) This does not imply that the blast loading on structures with windgirders may not differ from the loading without such a girder. Information on blast loading of tanks with windgirders is not yet available.

Consider a storage tank with a windgirder attached to the tank at a height $\bar{L}$ from the tank bottom, Fig. VI-1. If the frequencies and mode shapes of an empty or partially full cylindrical tank without a windgirder are known, an expression for the frequencies of the stiffened tank is derived as follows. Rayleigh's Principle, $T_{max} = V_{max}$, can be written for the n^{th} mode of vibration in the form:

$$T_{ns} + T_{nR} = V_{ns} + V_{nR} \qquad \text{(VI-1)}$$

where T_{ns} and V_{ns} are the maximum values of the kinetic energy and potential energy respectively of the empty or partially filled unstiffened shell and T_{nR} and V_{nR} are the maximum values of the kinetic energy and potential energy of the windgirder. Defining the quantity $\bar{T}_n$ as in Eq. (V-39), the frequency of a shell with a windgirder is obtained from the relation:

$$\omega_n^2 = \frac{V_{ns} + V_{nR}}{\bar{T}_{ns} + \bar{T}_{nR}} \quad . \qquad \text{(VI-2)}$$

By definition, $\bar{T}_{ns} = \frac{1}{2} M_{ns}$ and therefore $V_{ns} = \frac{1}{2} M_{ns}\omega^2_{ns}$, where M_{ns} and ω_{ns} are the generalized mass in the n^{th} mode and the frequency in the n^{th} mode respectively of a shell without a windgirder. Substituting the above expressions into Eq. (VI-2), the frequency ω_n is given by:

$$\omega^2_n = \frac{\omega^2_{ns} + \dfrac{2V_{nR}}{M_{ns}}}{1 + \dfrac{M_{nR}}{M_{ns}}} \qquad \text{(VI-3)}$$

where M_{nR} is the generalized mass of the windgirder.

Assuming that the shapes of the modes of the tank with the wind-girder are the same as the shapes of the unstiffened tank, the windgirder ring must undergo the displacements $w(\bar{L}, \theta)$ and $v(\bar{L}, \theta)$, of the tank at the height $z = \bar{L}$;

$$v(\bar{L}, \theta) = C_n \left[\frac{V_n}{W_n} \quad \frac{\bar{L}}{a} \right] \sin n\theta \qquad \text{(VI-4)}$$

$$w(\bar{L}, \theta) = C_n \left[\frac{Y_n}{W_n} + \left(\frac{2\bar{L} - L}{2a}\right) \right] \cos n\theta \qquad \text{(VI-5)}$$

where C_n, the normalization coefficient for the unstiffened tank and the mode shape ratios $\frac{V_n}{W_n}$ and $\frac{Y_n}{W_n}$ are given in Section (I) or Section (V).
The potential energy of the ring, V_R, in terms of the ring displacements v and w is:

$$V_R = \frac{E_R I_R}{2a^3} \int_0^{2\pi} (w_{\theta\theta} + w)^2 \, d\theta + \frac{E_R A_R}{2a} \int_0^{2\pi} (v_\theta - w)^2 \, d\theta \qquad \text{(VI-6)}$$

which, upon substitution of Eq. (VI-4)-(VI-5) becomes

$$V_{nR} = \frac{E_R I_R \pi C_n^2}{2a^3} \left[\frac{Y_n}{W_n} + \left(\frac{2\bar{L} - L}{2a}\right)^2 \right] (1 - n^2)^2$$

$$+ \frac{E_R A_R \pi C_n^2}{2a} \left[\frac{V_n}{W_n} \frac{\bar{L}n}{a} - \frac{Y_n}{W_n} - \left(\frac{2\bar{L} - L}{2a}\right) \right]^2 \qquad \text{(VI-7)}$$

The generalized mass of the ring, M_{nR}, is

$$M_{nR} = 2\bar{T}_R = \rho_R A_R \int_0^{2\pi} (v^2 + w^2) \, a \, d\theta \qquad \text{(VI-8)}$$

which, upon substitution of Eq. (VI-4)-(VI-5) becomes:

$$M_{nR} = \rho_R A_R \pi a\, C_n^2 \left\{ \left(\frac{V_n}{W_n}\frac{\bar{L}}{a}\right)^2 + \left[\frac{Y_n}{W_n} + \left(\frac{2\bar{L} - L}{2a}\right)\right]^2 \right\} \qquad \text{(VI-9)}$$

Since in all cases, the modes of vibration of the unstiffened shell have been normalized to $M_i = 2\pi aLm_i$, the total mass of the empty tank, the generalized mass of the shell in all modes "n" is given by:

$$M_{ns} = 2\pi aLm_i \qquad \text{(VI-10)}$$

Once the frequency of the unstiffened shell, ω_{ns}, is known, Eq. (VI-7)-(VI-10) can be substituted into Eq. (VI-3) and the value of ω^2_n can be computed.

Table (VI-1) contains the values of the frequencies ω_n and ω_{ns} for the typical case of an empty steel shell, 100 ft. in diameter and 40 ft high, having the windgirder section shown in Fig. (VI-2).(3) The windgirder has been attached to the shell at a height $\bar{L}$ = 39 ft. It is seen that the correction due to the windgirder is very small; this correction is less than 5 1/2 per cent in all modes. The situation in fluid filled tanks is quite similar. Therefore, it is concluded that for the type of storage tanks under consideration, the effect of the windgirder may be neglected and the formulas of Section (I)-(V) can be used even for structures with windgirders.(4)

(3) American Petroleum Institute Specification for Welded Oil Storage Tanks, Fifteenth Edition, March 1958, Pg. 13 and 70: The minimum required section modulus for a stiffening ring is $Z = 0.0001\, D^2 H_2$ where Z is the section modulus in in^3, D is the nominal diameter of the tank in feet, and H_2 is the total height of the tank shell in feet. For a tank of 100 feet diameter and 40 feet height, $Z = 40\ in^3$. The section modulus of the windgirder shown in Fig. (VI-2) is 47.7 in^3 (See Pg. 70, A.P.I. Specs).

(4) See Footnote 2, Section VI.

Table VI-1

Effect of Windgirder on the Frequency of Cylindrical Tanks

Empty Steel Tank

$L = 40$ ft., $a = 50$ ft., $\bar{L} = 39$ ft., $A_R = 11.2$ in^2, $I_R = 449$ in^4

n	ω_{ns} (rad/sec)	ω_n (rad./sec)	% Difference
1	217.4	211.5	2.7%
2	140.3	134.2	4.4%
3	94.2	89.4	5.1%
4	66.8	63.4	5.2%
5	49.9	47.9	4.1%

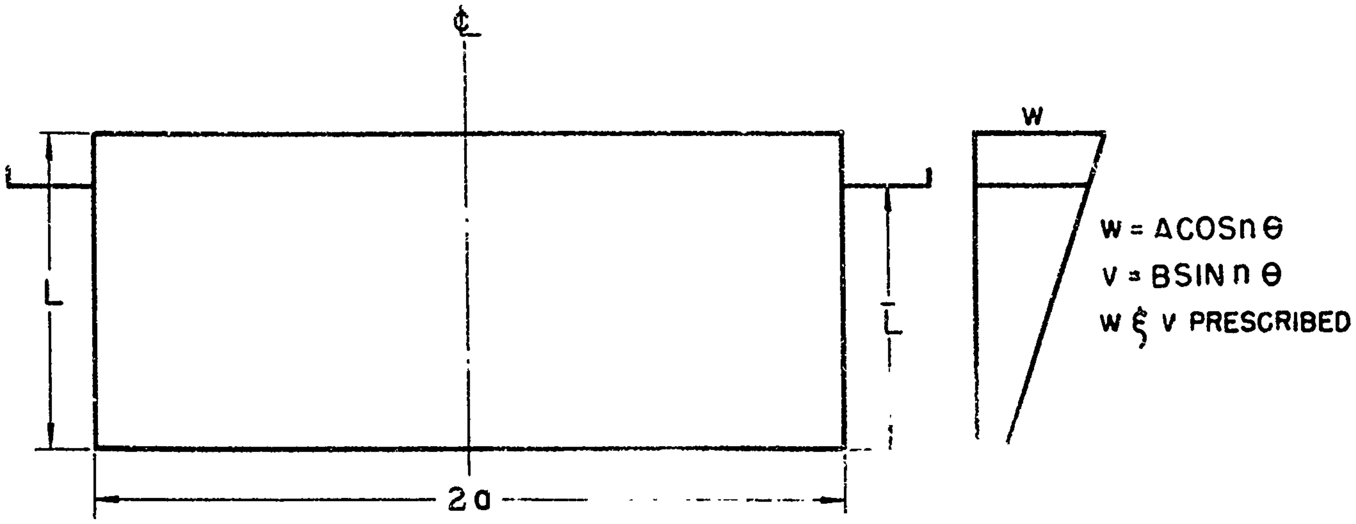

FIG. VI-1 TANK WITH WIND GIRDER

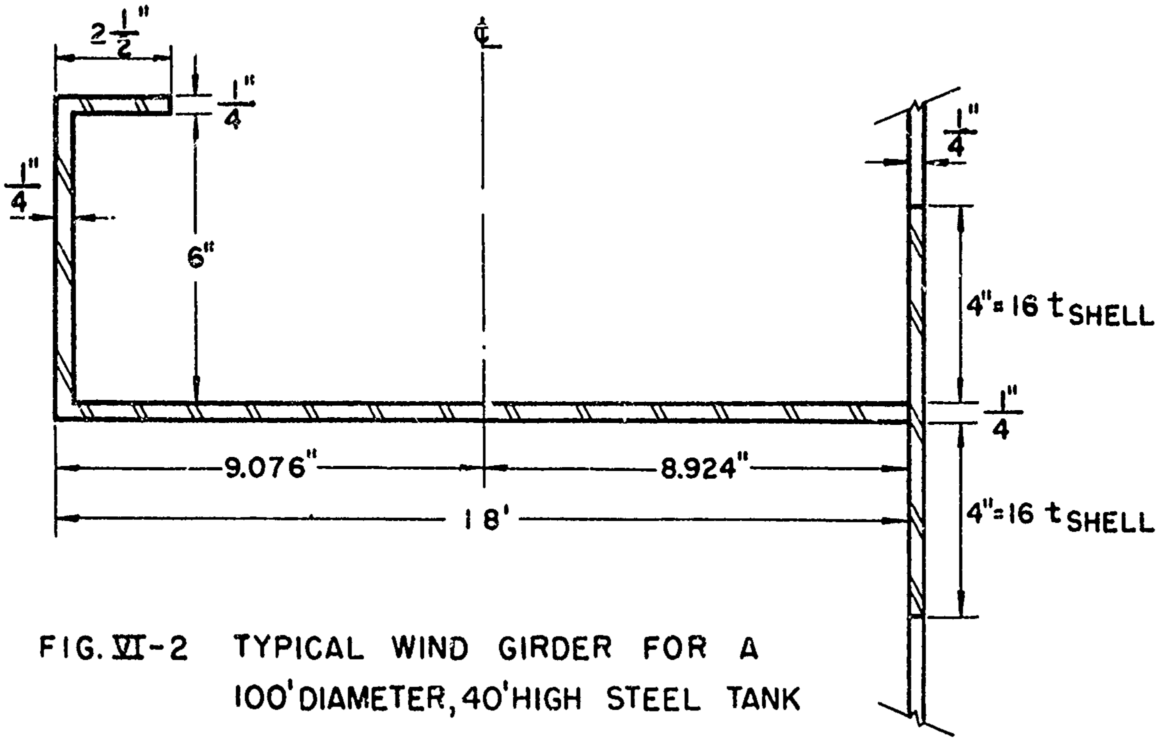

FIG. VI-2 TYPICAL WIND GIRDER FOR A 100' DIAMETER, 40' HIGH STEEL TANK

VII Forced Vibrations of Empty and Partially Full Cylindrical Fuel Tanks.

The responses of empty and partially (up to 9/10 of the height) full cylindrical tanks to dynamic loading are analyzed in this Section. For empty shells, the blast loading consists of a radial pressure dynamically applied to the shell, while for partially full shells, the vertical pressures of the blast on the surface of the liquid must also be considered. Information regarding the pressure history is given in the various Armour Institute[1] reports and specific methods for the utilization of the pressure data are developed in the present paper. The cases of a partially full tank and an empty tank will be separately treated.

Using the modes of free vibrations of the empty or partially full tank respectively as generalized coordinates, the response can be expressed in terms of the infinite number of modes. The required frequencies and modes of empty and partially full cylindrical shells were determined in Sections (I) and (V) of the report respectively. For each integral value "n" of circumferential waves in the shell displacement, there exists an infinite number of frequencies ω_{nj}, $j = 1, 2, 3. . . .$ and corresponding mode shapes $u_{nj}(z, \theta)$, $v_{nj}(z, \theta)$ and $w_{nj}(z, \theta)$. Calling $q_{nj}(t)$ the generalized coordinate corresponding to the mode "nj", the response of the shell can be written in terms of a summation of the normal modes:

$$u(z, \theta, t) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} u_{nj}(z, \theta) q_{nj}(t) \qquad \text{(VII -1)}$$

(1) Blast Effects on Storage Tank Type Structures, Armour Research Foundation Report Number

$$v(z, \theta, t) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} v_{nj}(z, \theta) q_{nj}(t) \qquad \text{(VII -2)}$$

$$w(z, \theta, t) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} w_{nj}(z, \theta) q_{nj}(t) \qquad \text{(VII -3)}$$

When determining the frequency numbers M_{nj} (which are proportional to ω_{nj}^2) it is found that the values M_{n1} are smaller-usually very much smaller, than unity, while the values of M_{nj} for $j > 1$ are larger than unity. The modes characterized by $j > 1$ represent primarily extensional high frequency oscillations of the shell. For loads of moderately long duration compared to the fundamental period T_{n1} for each value of n, these high frequency modes give only small contributions to the response of the shell. The major response of the structure can be determined from the modes characterized by $j = 1$ and all modes for which $j > 1$ are therefore dropped from the summation.

The special case $n = 0$ has been discussed in Sections (I-e) and(V-f) of the report. For the empty shell, an infinite number of frequency numbers M_{oj} are found to lie between closely spaced limits, thus making a modal analysis for the case $n = 0$ impracticable. A suitable alternative method to obtain the contribution of the $n = 0$ term is to consider the shell acting as a series of separate rings, each having a purely radial displacement $w_o(t)$. For the fluid filled shell, however, the displacements of the rings on different levels are coupled by the fluid and a spectrum of gradually increasing frequencies is again obtained. The fundamental frequency $j = 1$

in the filled shell contains again the major contribution to the response for $n = 0$, as in the case where $n \neq 0$.

The analysis of the response of the shell in the mode $n = 0$ will be discussed separately in Part (C) of the present section.

For the pressure loadings of moderately long duration under consideration, the response of the shell can therefore be written in the form

$$u(z, \theta, t) = \sum_{n=1}^{\infty} u_n(z, \theta) q_n(t) \tag{VII-4}$$

$$v(z, \theta, t) = \sum_{n=1}^{\infty} v_n(z, \theta) q_n(t) \tag{VII-5}$$

$$w(z, \theta, t) = w_o(z) q_o(t) + \sum_{n=1}^{\infty} w_n(z, \theta) q_n(t) \tag{VII-6}$$

where the functions u_n, v_n and w_n are the modes of free vibrations of the empty or the partially full shell corresponding to $j = 1$. Both the five constant and the three constant approximations of Section I will be considered in the following analysis. It should be noted that for cases in which shell strains and stresses are required, the five constant analysis should be used. Expressions will now be developed for the determination of the generalized coordinates $q_n(t)$ from the information given in the Armour Report.

a) Blast Pressures on Fuel Tanks - Nomenclature and Measuring Arrangement

The dynamic load information required to determine the response of a cylindrical fuel tank under blast loading is obtained from a report on shock tube experiments by the Armour Research Foundation (See footnote 1, Section 7). A description of the shock tube tests is given in the Armour report. The gages were arranged to give enough pressure information to allow for a meaningful forced vibration analysis of the tank. Two types of pressures were required; a) the radial pressures on the tank walls, and b) the vertical pressure on the surface of the fluid filling. In this section, the nomenclature and the arrangement of the pressure information required for a forced vibration analysis will be described.

For purpose of obtaining vertical pressures on the fluid surface of the tank, the surface is divided into sixteen sectors by means of eight diametral planes spaced 22 1/2 degrees apart. Each sector is denoted as S_k, k = 1-16 and is located by an angle θ measured to the midpoint of sector k. The intersection of the bisector of S_k and the periphery of the shell is called i, i = 0-15. The angle θ is measured positive clockwise from the zero position which is the point at which the shock wave first hits the shell. The sector S_k is bounded by the angles θ_k and θ_{k+1}.

Each sector is further divided radially into three subareas S_{kp}, where the subscripts p = 1, 2, 3 refer to the outermost, middle and interior subareas of the sector, respectively. Each subarea contains pressure gages from which a mean vertical pressure P_{kp} is obtained. The pressure P_{kp} is considered constant over the subarea S_{kp}. Fig. (VII-2)

and (VII-3) show the divisions of the fluid surface and the nomenclature used in the following sections of the report.

For purposes of obtaining radial pressures on the walls of the tank, gages were placed to measure the average pressure along a generator denoted by i at the various heights shown in Fig. (VII-1). The outside wall pressure are thus obtained for a sector S_k in terms of the four pressures P_{i1}-P_{i4}. For the radial pressure on the inner walls of the shell, the pressure P_{i6} is used for a 9/10 full tank and the pressures P_{i5} and P_{i6} are used for a half full tank. In each subarea shown in Fig. (VII-1), the pressure P_{ih}, h = 1-6, is considered constant over the area.

Once the weapon yield and the peak ambient overpressure are selected, the external pressures acting on the tank can be obtained, and the forced vibration analysis of the following sections can be made.

b) n ≠ 0 Five Constant Approximation - Partial Filling $0 < \gamma < 1$. (See Fig. VII-5)

Substituting the expression Eq. (V-15) to (V-17) for the mode shapes into Eq. (VII-4)-(VII-6) for the response, the displacement u, v, w of a point on the shell becomes:

$$u(z, \theta, t) = \sum_{n=1}^{\infty} C_n \left[\frac{U_n}{W_n} \frac{z}{a} + \frac{X_n}{W_n} \left(\frac{z^2}{a^2} - \frac{3Lz}{4a^2} \right) \right] \cos n\theta \, q_n(t) \quad \text{(VII-7)}$$

$$v(z, \theta, t) = \sum_{n=1}^{\infty} C_n \frac{V_n}{W_n} \frac{z}{a} \sin n\theta \, q_n(t) \quad \text{(VII-8)}$$

$$w(z, \theta, t) = w_o(z) q_o(t) + \sum_{n=1}^{\infty} C_n \left[\frac{Y_n}{W_n} + \left(\frac{z}{a} - \frac{L}{2a} \right) \right] \cos n\theta \, q_n(t). \quad \text{(VII-9)}$$

Each mode is normalized to the total mass of the empty tank, $\bar{M}_1 = 2\pi a L m_1$ and the normalization coefficients C_n, are given by Eq. (V-19).

The generalized coordinates $q_n(t)$ satisfy Lagrange's equations

$$\ddot{q}_n + \omega_n^2 q_n = \frac{Q_n}{\bar{M}_1} = \frac{Q_n}{2\pi a L m_1} \quad \text{(VII-10)}$$

where ω_n , the frequency of vibrations of the tank in the n^{th} mode is computed from the lowest root of the frequency determinant Eq. (V-13), [or Eq. (V-25) if bending effects are small such that the shell can be considered to act as a membrane] and $Q_n(t)$ is the generalized force in the n^{th} mode.

The generalized force, $Q_n(t)$, may be written as the sum of two terms:

$$Q_n(t) = Q_{n1}(t) + Q_{n2}(t) \quad \text{(VII-11)}$$

where $Q_{n1}(t)$ is the generalized force produced by the radial blast

pressure on the tank walls and $Q_{n2}(t)$ is the generalized force produced by the vertical blast pressures acting on the surface of the tank filling. The force $Q_{n1}(t)$ is given by

$$Q_{n1}(t) = \int_0^{2\pi} \int_0^{L} P(\theta, z, t)\, w_n(\theta, z)\, a d\theta dz \quad . \tag{VII-12}$$

Substituting

$$w_n(z, \theta) = C_n \left[\frac{z}{a} + \left(\frac{Y_n}{W_n} - \frac{L}{2a} \right) \right] \cos n\theta \tag{VII-13}$$

into Eq. (VII-12), $Q_{n1}(t)$ may be written in the form

$$Q_{n1}(t) = \int_0^{2\pi} \left\{ \int_0^{L} P(\theta, z, t)\, C_n \left[\frac{z}{a} + \left(\frac{Y_n}{W_n} - \frac{L}{2a} \right) \right] dz \right\} \cos n\theta \; a d\theta \tag{VII-14}$$

Referring to Section (VII-a) in which the pressure measurements made on the Armour Shock Tube model are described in detail, it is evident that the spatial integration with respect to z can be performed at each of the sixteen vertical generators "i" on the shell, $i = 0, 1, 2, \ldots 16$ as shown in Figure (VII-1).

Denoting the integral for each i as f_{ni},

$$f_{ni} = \int_0^{L} P_i(z, t)\, C_n \left[\frac{z}{a} + \left(\frac{Y_n}{W_n} - \frac{L}{2a} \right) \right] dz \tag{VII-15}$$

and letting P_{ih} be the pressure at the various heights h of the tank ($h = 1, 2, 3, 4, 5, 6$) in the segment S containing the point i, the function f_{ni} is easily computed in terms of the P_{ih}. For example, for the case of a 9/10 full tank, $\gamma = 0.9$, $f_{ni}^{(2)}$ becomes

$$f_{ni} = C_n a \xi^2 \left[-.105P_{i1} - .015P_{i2} + .075P_{i3} + .045(P_{i4} - P_{i6}) \right]$$

$$+ C_n a \;\; .3\xi \frac{Y_n}{W_n} \left[P_{i1} + P_{i2} + P_{i3} + 0.333\,(P_{i4} - P_{i6}) \right] \tag{VII-16}$$

Equation (VII-14) may be written in the form:

$$Q_{n1}(t) = \int_o^{2\pi} f_n(\theta, t) \cos n\theta \quad a d\theta \qquad \text{(VII-17)}$$

where $f_n(\theta, t)$ is a continuous function of the angle θ. The function $f_n(\theta, t)$ is expressed as a Fourier Cosine Series,

$$f_n(\theta, t) = a_{no}(t) + \sum_{j=1}^{8} a_{nj}(t) \cos j\theta \qquad \text{(VII-18)}$$

where the coefficients a_{no} and a_{nj} are determined so that $f_n(\theta, t) = f_{ni}$ when $\theta = \theta_i$ ($i = 1, 2, \ldots 16$.) This requirement restricts the series to nine terms and the coefficients a_{nj} (3) are

$$a_{no} = \frac{1}{16} \sum_{i=0}^{15} f_{ni}(\frac{\pi i}{8}) \qquad \text{(VII-19)}$$

$$a_{nj} = \frac{1}{8} \sum_{i=0}^{15} f_{ni} (\frac{\pi i}{8}) \cos j\frac{\pi i}{8} \qquad \text{(VII-20)}$$

$$j = 1, 2, 3, \ldots .7$$

$$a_{n8} = \frac{1}{16} \sum_{i=0}^{15} f_{ni}(\frac{\pi i}{8}) \cos \pi i \qquad \text{(VII-21)}$$

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(2) For the case of a 5/10 full tank, the formula for f_{ni} becomes

$$f_{ni} = C_n a \ell^2 \left[- .105P_{11} - .015P_{12} + .075P_{13} + .045(P_{14} - P_{16}) - .08P_{15}\right]$$

$$+ C_n a \; .3\ell \frac{Y_n}{W_n} \left[P_{11} + P_{12} + P_{13} + 0.333(P_{14} - P_{16}) - 1.333P_{15}\right]$$

(3) The harmonic analysis in which a Fourier Series is passed through the sixteen points i and where the coefficients a_{nj} are determined is given in the Appendix to Section VII.

Substituting Eq. (VIII-18) into Eq. (VII-17),

$$Q_{n1}(t) = \int_0^{2\pi} \left[a_{no} + \sum_{j=1}^{8} a_{nj}(t) \cos j\theta \right] \cos n\theta \quad a d\theta \tag{VII-22}$$

and using the orthogenality relation

$$\int_0^{2\pi} \cos j\theta \cos n\theta \, d\theta = \begin{cases} 0 & n \neq j \\ \pi & n = j \end{cases} \tag{VII-23}$$

the generalized force $Q_{n1}(t)$ becomes

$$Q_{n1}(t) = \pi a \, a_{nn}(t) \quad . \qquad n = 1, 2, \ldots 8 \tag{VII-24}$$

The load information given in the Armour report is sufficient to give generalized forces in the nine modes, n = 0 to n = 8. In general therefore, a nine mode elastic analysis may be made for the determination of the response of the shell; in practice, with loads of moderately long duration being considered, the series can usually be stopped after the fourth to sixth mode.

The generalized force $Q_{n2}(t)$ due to the vertical pressure acting on the fluid surface at $z = \gamma L$ is given by (4)

$$Q_{n2}(t) = -\int_0^{2\pi} \int_0^{a} P(r, \theta, t) \zeta_n(r,\theta) r dr d\theta \tag{VII-25}$$

where $\zeta_n(r,\theta)$ is the space dependent part of $\zeta_n(r, \theta, t)$ which is the displacement of the fluid surface at $z = \gamma L$. From Eq. (IV-110)-(IV-112):

$$\zeta_n(r, \theta) = \sum_{i=1}^{\infty} \zeta_{ni}(r,\theta) = \frac{C_n \gamma L}{a} \sum_{i=1}^{\infty} \bar{\beta}_{ni} J_n\left(\frac{\alpha_{ni} r}{a}\right) \cos n\theta \tag{VII-26}$$

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(4) The minus sign is due to the fact that the pressure $P(r,\theta, t)$ is considered positive when acting downward, while positive displacements $\zeta_n(r, \theta)$ are upwards.

where

$$\bar{\beta}_{ni} = \frac{2\alpha_{ni}\tanh\left(\frac{\alpha_{ni}\gamma L}{a}\right)}{(\alpha_{ni}^2 - n^2)J_n(\alpha_{ni})}\left[G_n + \frac{1}{2} - \frac{1}{2\alpha_{ni}\frac{\gamma L}{a}}\left(\frac{1}{\tanh\left(\frac{\alpha_{ni}\gamma L}{a}\right)} - \frac{1}{\sinh\left(\frac{\alpha_{ni}\gamma L}{a}\right)} + \tanh\left(\frac{\alpha_{ni}\gamma L}{2a}\right)\right)\right] \quad \text{(VII-27)}$$

Substituting Eq. (VII-26) into Eq. (VII-25), $Q_{n2}(t)$ becomes

$$Q_{n2}(t) = - \int_0^{2\pi}\int_0^a P(r, \theta, t)\; C_n \frac{\gamma L}{a}\left[\sum_{i=1}^{\infty}\bar{\beta}_{ni}J_n\left(\frac{\alpha_{ni}r}{a}\right)\right]\cos\theta \;\; r dr d\theta \quad \text{(VII-28)}$$

Referring again to Fig. (VII-2) and (VII-3) of Section (VII-a), the fluid surface is divided into circular sectors denoted by k, k = 1, 2, 3,...16. Moreover, each sector k is further subdivided into the subareas S_{kp}, p = 1, 2, 3, corresponding to the location of the roof pressure measuring gages. In each subarea

$$S_{kp}\begin{cases} k = 1, 2, \ldots 16 \\ p = 1, 2, 3 \end{cases} \quad \text{(VII-29)}$$

the pressure is considered constant and the integration of Eq. (VII-28) can be performed analytically in each subarea. The expression for the generalized force $Q_{n2}(t)$ becomes therefore

$$Q_{n2}(t) = \sum_{k=1}^{16}\sum_{p=1}^{3}\int\int_{S_{kp}} - P_{kp}\frac{C_n\gamma L}{a}\left[\sum_{i=1}^{\infty}\bar{\beta}_{ni}J_n\left(\frac{\alpha_{ni}r}{a}\right)\right]\cos n\theta \; r dr \; d\theta \;. \quad \text{(VII-30)}$$

The contribution to $Q_{n2}(t)$ from the vertical pressures acting on the

subarea S_{kp} in Eq. (VII-30) can be written in the form

$$Q_{n2_{kp}} = -P_{kp} C_n \frac{\gamma L}{a} \int_{\theta_k}^{\theta_{k+1}} \cos n\theta d\theta \int_{r_1}^{r_2} \left[\sum_{i=1}^{\infty} \bar{\beta}_{ni} J_n\left(\frac{\alpha_{ni} r}{a}\right) \right] r dr \quad \text{(VII-31)}$$

where the limits on the integrals are determined by the particular subarea S_{kp}. Interchanging the integration with respect to r and the summation over i, and defining the quantities

$$I_{nip} = \int_{r_1}^{r_2} J_n\left(\frac{\alpha_{ni} r}{a}\right) r dr \quad \text{(VII-32)}$$

$$\psi_{nk} = \frac{\sin n\theta_{k+1} - \sin n\theta_k}{n} , \quad \text{(VII-33)}$$

the generalized force $Q_{n2_{kp}}$ becomes:

$$Q_{n2_{kp}} = - P_{kp} C_n \frac{\gamma L}{a} \psi_{nk} \sum_{i=1}^{\infty} \bar{\beta}_{ni} I_{nip} \quad . \quad \text{(VII-34)}$$

Analytical expressions for the integrals I_{nip} [Eq. (VII-32] are given in Table (VII-1) for n = 0(1)6.

The generalized force $Q_{n2}(t)$ is given by

$$Q_{n2}(t) = \sum_{k=1}^{16} \sum_{p=1}^{3} Q_{n2_{kp}}(t) \quad \text{(VII-35)}$$

which, upon substitution of Eq. (VII-34) becomes:

$$Q_{n2}(t) = \sum_{k=1}^{16} \sum_{p=1}^{3} -P_{kp} C_n \frac{\gamma L}{a} \psi_{nk} \left[\sum_{i=1}^{\infty} \bar{\beta}_{ni} I_{nip} \right] \quad . \quad \text{(VII-36)}$$

In practice, a sufficient number of the fluid modes, denoted by the subscript "i" must be considered, commensurate with the required accuracy.

For the type of study considered, the first three terms of the series of fluid modes suffice and the upper limit in the summation over i in Eq. (VII-36) becomes three.

The generalized force $Q_n(t)$ can be evaluated from Eq. (VII-11) and the generalized coordinates $q_n(t)$ can be computed by a numerical integration of Eq. (VII-10). A suitable numerical integration technique, due to Noumerov,(5) evaluates q_n by a forward step integration in time. Writing Eq. (VII-10) and the initial conditions for a system starting from rest,

$$\ddot{q}_n + \omega_n^2 q_n = \frac{Q_{n1}(t) + Q_{n2}(t)}{\bar{M}_i} = F_n(t) \qquad \text{(VII-37)}$$

$$q_n(0) = \dot{q}_n(0) = 0 \ .$$

Let "k" be the interval of the time steps, the recurrence formula for $q_n(t+k)$ becomes

$$q_n(t+k) = \frac{k^2}{12 + k^2\omega_n^2} \left[F_n(t-k) + 10F_n(t) + F_n(t+k) \right] + \left[\frac{24 - 10k^2\omega_n^2}{12 + k^2\omega_n^2} \right] q_n(t) - q_n(t-k) \qquad \text{(VII-38)}$$

Equation (VII-38) allows the determination of the generalized coordinate $q_n(t+k)$ in terms of known values at two previous time steps. The integration is started, using the formulas

$$q_n(0) = 0$$

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(5) Numerical Methods in Engineering by M. G. Salvadori and M. L. Baron, Prentice Hall, Second Printing, 1955, Pg. 118 ff

$$q_n(k) = \frac{k^2\left[10F(0) + F(k)\right]}{\left|24 + 2k^2\omega_n^2\right|} \qquad \text{(VII-39)}$$

Once the generalized coordinates $q_n(t)$ have been evaluated, the response of the shell is obtained from Eq. (VII- 7) (VII- 9). Using Eq. (I-30)-(I-32), the strains in the shell are given by the expressions[6]

$$a\epsilon_{\theta\theta} = a\epsilon_{\theta\theta}\Big]_{n=0} + \sum_{n=1}^{\infty} C_n\left[\frac{L}{2a} + \left(n\frac{V_n}{W_n} - 1\right)\frac{z}{a} - \frac{Y_n}{W_n}\right]\cos n\theta\, q_n(t) \qquad \text{(VII-40)}$$

$$a\epsilon_{zz} = \sum_{n=1}^{\infty} C_n\left[\frac{U_n}{W_n} + \left(\frac{2z}{a} - \frac{3}{4}\frac{L}{a}\right)\frac{X_n}{W_n}\right]\cos n\theta\, q_n(t) \qquad \text{(VII-41)}$$

$$a\epsilon_{z\theta} = \sum_{n=1}^{\infty} \frac{C_n}{2}\left[\frac{V_n}{W_n} - n\frac{z}{a}\frac{U_n}{W_n} - n\frac{X_n}{W_n}\left(\frac{z^2}{a^2} - \frac{3Lz}{4a^2}\right)\right]\sin n\theta\, q_n(t) \qquad \text{(VII-42)}$$

Using Eq. (VII-40)-(VII-42), the membrane shell stresses can be computed from the relations:

$$\sigma_{\theta\theta} = \frac{E}{a(1-\nu^2)}\left[a\epsilon_{\theta\theta} + \nu a\epsilon_{zz}\right] \qquad \text{(VII-43)}$$

$$\sigma_{zz} = \frac{E}{a(1-\nu^2)}\left[a\epsilon_{zz} + \nu a\epsilon_{\theta\theta}\right] \qquad \text{(VII-44)}$$

$$\sigma_{z\theta} = \frac{E}{a(1+\nu)}\left[a\epsilon_{z\theta}\right] \qquad \text{(VII-45)}$$

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(6) The strain $\epsilon_{\theta\theta}$ in the mode $n = 0$ is given by Eq. (VII- 73) in Part (c) of this section.

For the case of steel shells with concrete shielding, bending stresses may no longer be small enough to be ignored. Referring to Fig. (VII-4), the strains ϵ_{zz} and $\epsilon_{\theta\theta}$ are given by the relations

$$\epsilon_{zz} = \frac{\partial u}{\partial z} + (r - a) \frac{\partial^2 w}{\partial z^2} \tag{VII-46}$$

$$\epsilon_{\theta\theta} = \frac{1}{a} \frac{\partial v}{\partial \theta} + \left(\frac{r-a}{a}\right) \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} - \frac{w}{r} , \tag{VII-47}$$

Substituting Eq. (VII-7)-(VII-9) into Eq. (VII-46)-(VII-47), expressions for the strains in terms of the mode shapes and generalized coordinates can be obtained. The shell stress $\sigma_{\theta\theta}$ can then be evaluated from Eq. (VII-43).

c) $n = 0$ Partial Filling $0 < \gamma \leq 1$ (See Figure VII-5)

The major part of the response due to $n = 0$ terms is due to the mode with the lowest frequency, $j = 1$; similar to the situation for $n \neq 0$. Therefore only the one mode of frequency ω_o, is used. The radial displacement in this mode is

$$w = q_o(t) \tag{VII-48}$$

while $u(z, \theta, t) = v(z, \theta, t) = 0$. The frequency and the virtual mass of the fluid in the partially filled tank are given by Eq. (V-48), (V-49), and(IV-89), respectively:

$$\omega_o^2 = \frac{Eh}{(1-\nu^2)m_i a^2} \left[\frac{1}{1 + \frac{\gamma m_{vo}}{m_i}} \right] \tag{VII-49}$$

$$m_{vo} = \epsilon_o \rho a = \rho a \left[\frac{1}{4} + \frac{2}{3} \frac{\gamma^2 L^2}{a^2} - \sum_{i=1}^{\infty} \frac{2\tanh(\alpha_{oi}\frac{\gamma L}{a})}{(\alpha_{oi})^3 \frac{\gamma L}{a}} \right] \tag{VII-50}$$

The generalized coordinate $q_o(t)$ must satisfy Lagrange's equation:

$$\ddot{q}_o + \omega_o^2 q_o = \frac{Q_o}{\overline{M}_o} \tag{VII-51}$$

where $\overline{M}_o$, the generalized mass of the partially full tank in the mode $n = 0$, is given by

$$\overline{M}_o = m_i \int_0^L \int_0^{2\pi} a d\theta dz + m_{vo} \int_0^{\gamma L} \int_0^{2\pi} a d\theta dz = 2\pi a L m_i \left[1 + \gamma \frac{m_{vo}}{m_i} \right] \tag{VII-52}$$

The generalized force, $Q_o(t)$, may be written as the sum of two terms:

$$Q_o(t) = Q_{o1}(t) + Q_{o2}(t) \qquad \text{(VII-53)}$$

where $Q_{o1}(t)$ is the generalized force produced by the radial blast pressure on the tank walls and $Q_{o2}(t)$ is the generalized force produced by the vertical blast pressures acting on the surface of the tank filling. The force $Q_{o1}(t)$ is given by

$$Q_{o1}(t) = \int_0^{2\pi}\int_0^{L} P(\theta, z, t)\, a d\theta dz \qquad \text{(VII-54)}$$

Referring to Section (VII-a) and proceeding as in the case for $n \neq 0$, the integration of Eq. (VII-54) with respect to z can be performed at each of the sixteen vertical generators on the shell, $i = 0, 1, 2, \ldots 16$ as shown in Figure (VII-1).

Denoting the integral for each i as f_{oi},

$$f_{oi} = \int_0^{L} P_i(z, t)dz \qquad \text{(VII-55)}$$

and letting P_{ih} be the pressure at the various heights h of the tank ($h = 1, 2, 3, 4, 5, 6$) in the segment S containing the point i, the function f_{oi} is easily computed in terms of the P_{ih}. For example, for the case of a 9/10 full tank, $\gamma = 0.9$, f_{oi}[6] becomes

$$f_{oi} = .3a\xi \left[P_{i1} + P_{i2} + P_{i3} + .333(P_{i4} - P_{i6})\right] \qquad \text{(VII-56)}$$

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(6) For the case of a 5/10 full tank, the formula for f_{oi} becomes

$$f_{oi} = .3a\xi \left[P_{i1} + P_{i2} + P_{i3} + .333(P_{i4} - P_{i6}) - 1.333P_{i5}\right]$$

Equation (VII-54) may be written in the form

$$Q_{c1}(t) = \int_0^{2\pi} f_o(\theta, t)\, a d\theta \tag{VII-57}$$

where $f_o(\theta, t)$ is a continuous function of the angle θ. The function $f_o(\theta, t)$ can be expressed as a Fourier cosine series

$$f_o(\theta, t) = a_{oo}(t) + \sum_{j=1}^{8} a_{oj}(t)\cos j\theta \tag{VII-58}$$

where the coefficients a_{oo} and a_{oj} are determined so that $f_o(\theta, t) = f_{oi}$ when $\theta = \theta_i$ $(i = 1, 2, \ldots.16)$. It will be shown however, that only the first coefficient $a_{oo}(t)$ need be computed in order to determine $Q_{o1}(t)$. Substituting Eq. (VII-58) into Eq. (VII-57) and integrating, it is seen that only the coefficient a_{oo} appears and has to be computed. The generalized force $Q_{o1}(t)$ is

$$Q_{o1}(t) = 2\pi a\; a_{oo}(t) \tag{VII-59}$$

where the coefficient a_{oo} is obtained for $n = 0$ from Eq. (VII-19).

The generalized force $Q_{o2}(t)$ due to the vertical pressure acting on the fluid surface at $z = \gamma L$ is given by

$$Q_{o2}(t) = -\int_0^{2\pi}\int_0^{a} P(r, \theta, t)\, \mathcal{Z}_o(r)\, r dr d\theta \tag{VII-60}$$

where $\mathcal{Z}_o(r)$ is the space dependent part of $\mathcal{Z}_o(r, t)$, which is the displacement of the fluid surface at $z = \gamma L$. From Eq. (IV-91)-(IV-93):

$$\mathcal{Z}_o(r) = \frac{2\gamma L}{a} + \sum_{i=1}^{\infty} \mathcal{Z}_{oi}(r) = \frac{2\gamma L}{a} + \sum_{i=1}^{\infty} \beta_{oi} J_o\left(\frac{\alpha_{oi} r}{a}\right) \tag{VII-61}$$

where

$$\beta_{oi} = \frac{2\tanh(\alpha_{oi} \frac{\gamma L}{a})}{\alpha_{oi} J_o(\alpha_{oi})} \qquad \text{(VII-62)}$$

Substituting Eq. (VII-61) into Eq. (VII-60), $Q_{o2}(t)$ becomes

$$Q_{o2}(t) = - \int_0^{2\pi} \int_0^{a} P(r, \theta, t) \left[\frac{2\gamma L}{a} + \sum_{i=1}^{\infty} \beta_{oi} J_o\left(\frac{\alpha_{oi} r}{a}\right)\right] r dr d\theta . \text{(VII-63)}$$

Referring again to Fig. (VII-2) and (VII-3), the fluid surface is divided into the subareas S_{kp} as in the case $n \neq 0$. In each subarea, the pressure is considered constant and the integration of Eq. (VII-63) can be performed analytically in each subarea. The expression for $Q_{o2}(t)$ becomes therefore

$$Q_{o2}(t) = \sum_{k=1}^{16} \sum_{p=1}^{3} \int \int_{S_{kp}} - P_{kp} \left[\frac{2\gamma L}{a} + \sum_{i=1}^{\infty} \beta_{oi} J_o\left(\frac{\alpha_{oi} r}{a}\right)\right] r dr d\theta \qquad \text{(VII-64)}$$

The contribution to $Q_{o2}(t)$ from the vertical pressures acting on the subarea S_{kp} in Eq. (VII-64) can be written in the form

$$Q_{o2_{kp}} = -P_{kp} \int_{\theta_k}^{\theta_{k+1}} d\theta \int_{r_1}^{r_2} \left[\frac{2\gamma L}{a} + \sum_{i=1}^{\infty} \beta_{oi} J_o\left(\frac{\alpha_{oi} r}{a}\right)\right] r dr \quad \text{(VII-65)}$$

where the limits of the integration are determined by the particular subarea S_{kp}. Interchanging the integration with respect to r and the summation over i, and defining the quantity

$$I_{oip} = \int_{r_1}^{r_2} J_o \left(\frac{\alpha_{oi} r}{a}\right) r dr \qquad \text{(VII-66)}$$

the generalized force $Q_{o2_{kp}}$ becomes

$$Q_{o2_{kp}} = - P_{kp} (\theta_{k+1} - \theta_k) \left[\frac{\gamma L}{a} (r_2^2 - r_1^2) + \sum_{i=1}^{\infty} \beta_{oi} I_{oip} \right] . \quad \text{(VII-67)}$$

An analytical expression for I_{oip} is given in Table (VII-1).

The generalized force $Q_{o2}(t)$ is given by

$$Q_{o2}(t) = \sum_{k=1}^{16} \sum_{p=1}^{3} Q_{o2_{kp}} \quad \text{(VII-68)}$$

which, upon substitution of Eq. (VII-67) becomes

$$Q_{o2}(t) = \sum_{k=1}^{16} \sum_{p=1}^{3} - P_{kp}(\theta_{k+1} - \theta_k) \left[\frac{\gamma L}{a} (r_2^2 - r_1^2) + \sum_{i=1}^{\infty} \beta_{oi} I_{oip} \right] \quad \text{(VII-69)}$$

As in the case $n \neq 0$, a sufficient number of fluid modes must be considered commensurate with the required accuracy. For the type of study considered the first three terms[7] of the series of fluid modes suffice and the upper limit in the summation over i in Eq. (VII-69) becomes two.

The generalized force $Q_o(t)$ can be evaluated from Eq. (VII-53) and the generalized coordinate $q_o(t)$ can be computed by a numerical integration of Eq. (VII-51). Proceeding as in the case where $n \neq 0$, and using the initial conditions for a system starting from rest, the recurrence formula for $q_o(t+k)$ becomes

$$q_o(t+k) = \frac{k^2}{12 + k^2\omega_o^2} \left[F_o(t-k) + 10F_o(t) + F_o(t+k) \right] + \left[\frac{24 - 10k^2\omega_o^2}{12 + k^2\omega_o^2} \right] q_o(t) - q_o(t-k) \quad \text{(VII-70)}$$

(7) The constant displacement term is considered as the lowest fluid mode.

where

$$F_o(t) = \frac{Q_{o1}(t) + Q_{o2}(t)}{2\pi a L m_i \left[1 + \frac{\gamma m_{vo}}{m_i}\right]} \qquad \text{(VII-71)}$$

and "k" is the interval of the time steps.

The integration is started, using the formulas

$$q_o(0) = 0$$

$$q_o(k) = \frac{k^2 \left[10F(0) + F(k)\right]}{\left[24 + 2k^2\omega_o^2\right]} . \qquad \text{(VII-72)}$$

The generalized coordinate $q_o(t)$ gives the response of the shell in the mode $n = 0$. The corresponding hoop strain, $\epsilon_{\theta\theta}$ is given by

$$\epsilon_{\theta\theta} = - \frac{q_o(t)}{a} \qquad \text{(VII-73)}$$

and the hoop stress, $\sigma_{\theta\theta}$ becomes

$$\sigma_{\theta\theta} = \frac{E}{(1-\nu^2)} \epsilon_{\theta\theta} . \qquad \text{(VII-74)}$$

d) $n \neq 0$ Three Constant Approximation - Partial Filling

$0 < \gamma \leq 1$. (See Fig. VII-5)

Substituting the expression of Eq. (V-34)-(V-36) for the mode shapes into Eq. (VII-4)-(VII-6) for the response, the displacements u, v, w of a point on the shell become:

$$u(z, \theta, t) = \sum_{n=1}^{\infty} C_n \frac{U_n}{W_n} \frac{z}{a} \cos n\theta \; q_n(t) \qquad \text{(VII-75)}$$

$$v(z, \theta, t) = \sum_{n=1}^{\infty} C_n \frac{V_n}{W_n} \frac{z}{a} \sin n\theta \; q_n(t) \qquad \text{(VII-76)}$$

$$w(z, \theta, t) = w_o(z) q_o(t) + \sum_{n=1}^{\infty} C_n \frac{z}{a} \cos n\theta \; q_n(t) \qquad \text{(VII-77)}$$

Each mode is again normalized to the total mass of the empty tank, $\overline{M}_1 = 2\pi a L \, m_1$ and the normalization coefficients C_n are given by Eq. (V-37).

The generalized coordinates $q_n(t)$ satisfy Lagranges equations, Eq. (VII-10), where ω_n is computed from the lowest root of the frequency determinant, Eq. (V-33). The components Q_{n1} and Q_{n2} of the generalized force $Q_n(t)$ are obtained in the same manner as shown in Section (VII-b). For the present case, the integral f_{ni} for each point becomes

$$f_{ni} = \int_0^L P_i(z, t) \, C_n \frac{z}{a} \, dz \; . \qquad \text{(VII-78)}$$

Using Eq. (VII-78), expressions giving f_{ni} in terms of the pressures P_{ih} are obtained for the cases of 9/10 full and 1/2 full tanks:

$$f_{ni} = C_n a \xi^2 \left[.045 P_{i1} + .135 P_{i2} + .225 P_{i3} + .095 (P_{i4} - P_{i6}) \right] .$$

<u>9/10 full</u> (VII-79)

$$f_{ni} = C_n a \xi^2 \left| .045P_{i1} + .135P_{i2} + .225P_{i3} + .095(P_{i4} - P_{i6}) - .280P_{i5} \right|$$

1/2 full (VII-80)

Proceeding exactly as in Section (VII-b), the generalized force component $Q_{n1}(t)$ is obtained from Eq. (VII-24).

The generalized force $Q_{n2}(t)$ is given by Eq. (VII-25) where, using Eq. (IV-61a)-(IV-62),

$$\gamma_n(r, \theta) = \sum_{i=1}^{\infty} \gamma_{ni}(r, \theta) = \frac{C_n \gamma L}{a} \sum_{i=1}^{\infty} \beta_{ni} J_n\left(\frac{\alpha_{ni} r}{a}\right) \cos n\theta \quad \text{(VII-81)}$$

and β_{ni} is given by Eq. (IV-61b). Proceeding as in Section (VII-b), the generalized force component $Q_{n2}(t)$ is obtained from Eq. (VII-31). The numerical integration formulas, Eq. (VII-37)-(VII-39) can then be used to evaluate the generalized coordinate $q_n(t)$.

Once the coordinates $q_n(t)$ have been evaluated, the response of the shell is obtained from Eq. (VII-7)-(VII-9). Although the three constant approximation may be used to obtain an estimate of the response of the shell, it does not give accurate results for the strains and stresses in the shell. It must be emphasized that when shell strains and stresses are required, the five constant approximation must be used.

e) $n \neq 0$ Empty Shell - Five Constant and Three Constant Approximations

In Section (VII-b) and (VII-d), methods for the determination of the transient response of partially filled fuel tanks to blast loadings are developed. The methods presented in these sections are general and can be applied to the case of empty tanks, provided that the correct normalized modes and frequencies of empty tanks [Sec. (I-a) and (I-e)] are used in the analysis. Moreover, the generalized force component, $Q_{n2}(t)$, must be set equal to zero for the case of an empty tank. With these exceptions, the procedures of the previous sections can be used in determining the transient response of empty cylindrical fuel tanks to blasts.

Specific formulas in terms of the wall pressures P_{ih} cannot be given at this time, since the Armour report (See footnote 1, Section VII) contains only pressure information for 9/10 full and 1/2 full tanks.

f) n = 0, Empty Shell.

The response of empty shells for the special case n = 0 has been discussed in Section (I-e) and (VII-a). To obtain the contribution of the mode n = 0, the empty shell is considered to act as a series of separate rings[7]; each ring is subjected to the radial pressure P_{ih} acting over a circumferential strip of shell equal in depth to the depth of the vertical subarea "ih" [See Fig. (VII-1)] . The response of each ring,

$$w_{oh} = q_{oh}(t) \tag{VII-82}$$

satisfies Lagranges equations

$$\ddot{q}_{oh} + \omega_{oh}^2 \, q_{oh} = \frac{Q_{oh}(t)}{M_o} \tag{VII-83}$$

where ω_{oh}, the frequency of each ring, is given by

$$\omega_{oh}^2 = \frac{E}{\rho_i a^2} \tag{VII-84}$$

and $\overline{M}_o$, the generalized mass, is given by

$$\overline{M}_o = 2\pi a m_i \tag{VII-85}$$

The generalized force on each ring is denoted by $Q_{oh}(t)$ and can be obtained in a manner similar to that of Section (VII-c). Letting $P_{ih}(t)$ be the radial blast pressure at the various points i on the particular ring defined by the index h, the generalized force becomes

$$Q_{oh}(t) = \int_0^{2\pi} P_h(\theta, t) a d\theta \tag{VII-86}$$

(7) Each ring is identified by an index "h", corresponding to a particular subarea "ih", [See Fig. (VII-1)]

where

$$P_h(\theta, t) = a_{oo}(t) + \sum_{j=1}^{8} a_{oj}(t)\cos j\theta \cdot \qquad \text{(VII-87)}$$

The coefficients a_{oj}, $j = 0$-8, are determined so that $P_h(\theta, t) = P_{ih}(t)$ when $\theta = \theta_i$ ($i = 1, 2, \ldots 16$). Substituting Eq. (VII-87) into Eq. (VII-86) and integrating, it is seen that only the coefficient $a_{oo}(t)$ appears and has to be computed. The generalized force $Q_{oh}(t)$ is

$$Q_{oh}(t) = 2\pi a\, a_{oo}(t) \qquad \text{(VII-88)}$$

where the coefficient $a_{oo}(t)$ is given by

$$a_{oo}(t) = \frac{1}{16} \sum_{i=0}^{15} P_{ih}\left(\frac{\pi i}{8}\right) \qquad \text{(VII-89)}$$

Proceeding as in Section (VII-c), Eq. (VII-83) can be integrated using Eq. (VII-70) and (VII-72) where

$$F_o(t) = \frac{a_{oo}(t)}{m_i} \; . \qquad \text{(VII-90)}$$

The response $w_{oh}(t)$ is assumed to act at the center of each ring and the corresponding hoop strain and hoop stress for each ring can be evaluated using Eq. (VII-73)-(VII-74) respectively.

Appendix to Section VII - Determination of the Fourier Series Coefficients a_{nj}.

The continuous function $f_n(\theta, t)$ given by Eq. (VII-18),

$$f_n(\theta, t) = a_{no}(t) + \sum_{j=1}^{8} a_{nj}(t) \cos j\theta \tag{A-1}$$

is required for the determination of the generalized force component $Q_{n1}(t)$. The coefficients a_{nj} are chosen so that $f_n(\theta, t) = f_{ni}$ when $\theta = \theta_i$ ($i = 1$-16). In this section, the expressions for the coefficients a_{nj} in terms of the quantities f_{ni} will be developed.

Consider the Fourier Series

$$f_n(\theta) = a_{no} + \sum_{j=1}^{\infty} a_{nj} \cos j\theta \tag{A-2}$$

where the coefficients a_{nj} are given by the expressions

$$a_{nj} = \frac{\int_0^{2\pi} f_n(\theta) \cos j\theta \, d\theta}{\int_0^{2\pi} \cos^2 j\theta d\theta} \tag{A-3}$$

Let the value of the function $f_n(\theta)$ be known at N equally separated points i, i = 1-16, where each point i is located by the angle θ_i ($i = 1, 2, 3. . .N$). The coefficients a_{nj} of the series, Eq. (A-2), are to be evaluated so that $f_n(\theta) = f_n(\theta_i) = f_{ni}$ at each point i. Calling the interval of separation

$$\Delta\theta = \frac{2\pi}{N} \tag{A-4}$$

the coefficient of Eq. (A-3) can be written as a ratio of the summation over i,

$$a_{nj} = \frac{\sum_{i=1}^{N} f_{ni}(\theta_i) \cos j\theta_i}{\sum_{i=1}^{N} \cos^2 j\theta_i} \tag{A-5}$$

Setting $\theta_i = \frac{2\pi i}{N}$, $i = 0, 1, 2, \ldots N-1$, for convenience, Eq. (A-5) becomes:

$$a_{nj} = \frac{\sum_{i=0}^{N-1} f_{ni}\left(\frac{2\pi i}{N}\right)\cos j\,\frac{2\pi i}{N}}{\sum_{i=0}^{N-1} \cos^2 j\,\frac{2\pi i}{N}} \quad . \tag{A-6}$$

Using the summation formula

$$\sum_{i=0}^{N-1} \cos^2 \alpha i = \frac{N}{2} + \frac{\sin\alpha(2N-1) + \sin\alpha}{4\sin\alpha} \tag{A-7}$$

and setting $\alpha = \frac{2\pi}{N}\, j$, the following results are obtained:

$$j \neq 0,\ j \neq \frac{N}{2} : \quad \sum_{i=0}^{N-1} \cos^2 \frac{2\pi}{N}\, ji = \frac{N}{2} \tag{A-8}$$

$$j = 0 \qquad \sum_{i=0}^{N-1} \cos^2 0i = N \tag{A-9}$$

$$j = \frac{N}{2} \qquad \sum_{i=0}^{N-1} \cos^2 \pi i = N \quad . \tag{A-10}$$

Substituting Eq. (A-8)-(A-10) into Eq. (A-6), the coefficients a_{nj} become:

$$a_{nj} = \frac{2}{N} \sum_{i=0}^{N-1} f_{ni}\left(\frac{2\pi i}{N}\right) \cos j\,\frac{2\pi i}{N} \tag{A-11}$$

$$a_{no} = \frac{1}{N} \sum_{i=0}^{N-1} f_{ni}\left(\frac{2\pi i}{N}\right) \tag{A-12}$$

$$a_{n,\frac{N}{2}} = \frac{1}{N} \sum_{i=0}^{N-1} f_{ni}\left(\frac{2\pi i}{N}\right) \cos \pi i \tag{A-13}$$

For the particular case where $N = 16$, Eqs. (VII-19)-(VII-21) are obtained. It should be noted that when a harmonic analysis passing a Fourier Series through N points is made, the series terminates after the term $(\frac{N}{2} + 1)$.

Table VII-1 Evaluation of $I_{nip} = \int_{r_1}^{r_2} J_n(\frac{\alpha_{ni} r}{a}) r dr$

n = 0

$$\int_{r_1}^{r_2} r J_o(\frac{\alpha_{ni} r}{a}) dr = \left[\frac{ra}{\alpha_{ni}} \; J_1(\frac{\alpha_{ni} r}{a}) \right]_{r_1}^{r_2}$$

n = 1(*)

$$\int_{o}^{r} r J_1(\frac{\alpha_{ni} r}{a}) dr = - \frac{ra}{\alpha_{ni}} J_o(\frac{\alpha_{ni} r}{a}) + \frac{2a^2}{\alpha_{ni}^2} \sum_{n=0}^{\infty} J_{2n+1}(\frac{\alpha_{ni} r}{a})$$

n= 2

$$\int_{r_1}^{r_2} r \; J_2(\frac{\alpha_{ni} r}{a}) dr = \left[- \frac{2a^2}{\alpha_{ni}^2} \; J_o(\frac{\alpha_{ni} r}{a}) - \frac{ra}{\alpha_{ni}} J_1(\frac{\alpha_{ni} r}{a}) \right]_{r_1}^{r_2}$$

n = 3

$$\int_{o}^{r} r \; J_3(\frac{\alpha_{ni} r}{a}) dr = \frac{ra}{\alpha_{ni}} \; J_o \; (\frac{\alpha_{ni} r}{a}) - \frac{8a^2}{\alpha_{ni}^2} \; J_1(\frac{\alpha_{ni} r}{a}) + \frac{6a^2}{\alpha_{ni}^2} \sum_{n=0}^{\infty} J_{2n+1}(\frac{\alpha_{ni} r}{a})$$

n = 4

$$\int_{r_1}^{r_2} r \; J_4(\frac{\alpha_{ni} r}{a}) dr = \frac{a^2}{\alpha_{ni}^2} \left[8J_o(\frac{\alpha_{ni} r}{a}) + (\frac{\alpha_{ni}^2 r^2 - 24a^2}{\alpha_{ni} ar}) \; J_1 \; (\frac{\alpha_{ni} r}{a}) \right]_{r_1}^{r_2}$$

$$\int_{o}^{r} r \; J_4(\frac{\alpha_{ni} r}{a}) dr = \frac{a^2}{\alpha_{ni}^2} \left[8J_o(\frac{\alpha_{ni} r}{a}) + (\frac{\alpha_{ni}^2 r^2 - 24a^2}{\alpha_{ni} ar}) \; J_1(\frac{\alpha_{ni} r}{a}) + 4 \right]$$

n = 5

$$\int_{o}^{r} r \; J_5(\frac{\alpha_{ni} r}{a}) dr = \frac{a^2}{\alpha_{ni}^2} \left[- \frac{64aJ_2(\frac{\alpha_{ni} r}{a})}{\alpha_{ni} r} + 8J_1(\frac{\alpha_{ni} r}{a}) - \frac{\alpha_{ni} r}{a} J_o(\frac{\alpha_{ni} r}{a}) \right.$$

$$\left. + \; 10 \sum_{n=0}^{\infty} \; J_{2n+1}(\frac{\alpha_{ni} r}{a}) \right]$$

n = 6

$$\int_{r_1}^{r_2} r\, J_6\left(\frac{\alpha_{ni} r}{a}\right) dr = \frac{a^2}{\alpha_{ni}^2} \left| - \frac{480 a^2\, J_2\left(\frac{\alpha_{ni} r}{a}\right)}{\alpha_{ni}^2 r^2} + \left(\frac{144 a^2 - \alpha_{ni}^2 r^2}{\alpha_{ni} a r}\right) J_1\left(\frac{\alpha_{ni} r}{a}\right) - 18\, J_o\left(\frac{\alpha_{ni} r}{a}\right) \right]_{r_1}^{r_2}$$

$$\int_{o}^{r} r\, J_6\left(\frac{\alpha_{ni} r}{a}\right) dr = \frac{a^2}{\alpha_{ni}^2} \left[- \frac{480 a^2 J_2\left(\frac{\alpha_{ni} r}{a}\right)}{\alpha_{ni}^2 r^2} + \left(\frac{144 a^2 - \alpha_{ni}^2 r^2}{\alpha_{ni} a r}\right) J_1\left(\frac{\alpha_{ni} r}{a}\right) - 18\, J_o\left(\frac{\alpha_{ni} r}{a}\right) + 6 \right] .$$

n = 7

$$\int_{o}^{r} r\, J_7\left(\frac{\alpha_{ni} r}{a}\right) dr = \frac{a^2}{\alpha_{ni}^2} \left[\left(- \frac{9216 a^4}{\alpha_{ni}^4 r^4} + \frac{1664 a^2}{\alpha_{ni}^2 r^2} - 32 \right) J_1\left(\frac{\alpha_{ni} r}{a}\right) + \left(\frac{4608 a^3}{\alpha_{ni}^3 r^3} - \frac{256 a}{\alpha_{ni} r} + \frac{\alpha_{ni} r}{a} \right) J_o\left(\frac{\alpha_{ni} r}{a}\right) + 14 \sum_{n=0}^{\infty} J_{2n+1}\left(\frac{\alpha_{ni} r}{a}\right) \right]$$

n = 8

$$\int_{r_1}^{r_2} r\, J_8\left(\frac{\alpha_{ni} r}{a}\right) dr = \frac{a^2}{\alpha_{ni}^2} \left[\left(- \frac{107520 a^5}{\alpha_{ni}^5 r^5} + \frac{21120 a^3}{\alpha_{ni}^3 r^3} - \frac{480 a}{\alpha_{ni} r} + \frac{\alpha_{ni} r}{a} \right) J_1\left(\frac{\alpha_{ni} r}{a}\right) + \left(\frac{53760 a^4}{\alpha_{ni}^4 r^4} - \frac{3840 a^2}{\alpha_{ni}^2 r^2} + 32 \right) J_o\left(\frac{\alpha_{ni} r}{a}\right) \right]_{r_1}^{r_2}$$

n = 8 (Cont.)

$$\int_0^r r\, J_8\left(\frac{\alpha_{ni} r}{a}\right) dr = \frac{a^2}{\alpha_{ni}^2}\left[\left(-\frac{107520a^5}{\alpha_{ni}^5 r^5} + \frac{21120a^3}{\alpha_{ni}^3 r^3} - \frac{480a}{\alpha_{ni} r} + \frac{\alpha_{ni} r}{a}\right) J_1\left(\frac{\alpha_{ni} r}{a}\right) + \right.$$

$$\left. + \left(\frac{53760a^4}{\alpha_{ni}^4 r^4} - \frac{3840a^2}{\alpha_{ni}^2 r^2} + 32\right) J_o\left(\frac{\alpha_{ni} r}{a}\right) + 8\right]$$

(*) For n = odd integers, the integral is obtained as an infinite series in $J_{2n+1}(\alpha_{ni}\ r/a)$ and a number of terms commensurate with the required accuracy must be taken. Moreover, the integral between limits r_1 and r_2 is computed from:

$$\int_{r_1}^{r_2} f(r)\, dr = \int_o^{r_2} f(r) dr - \int_o^{r_1} f(r) dr.$$

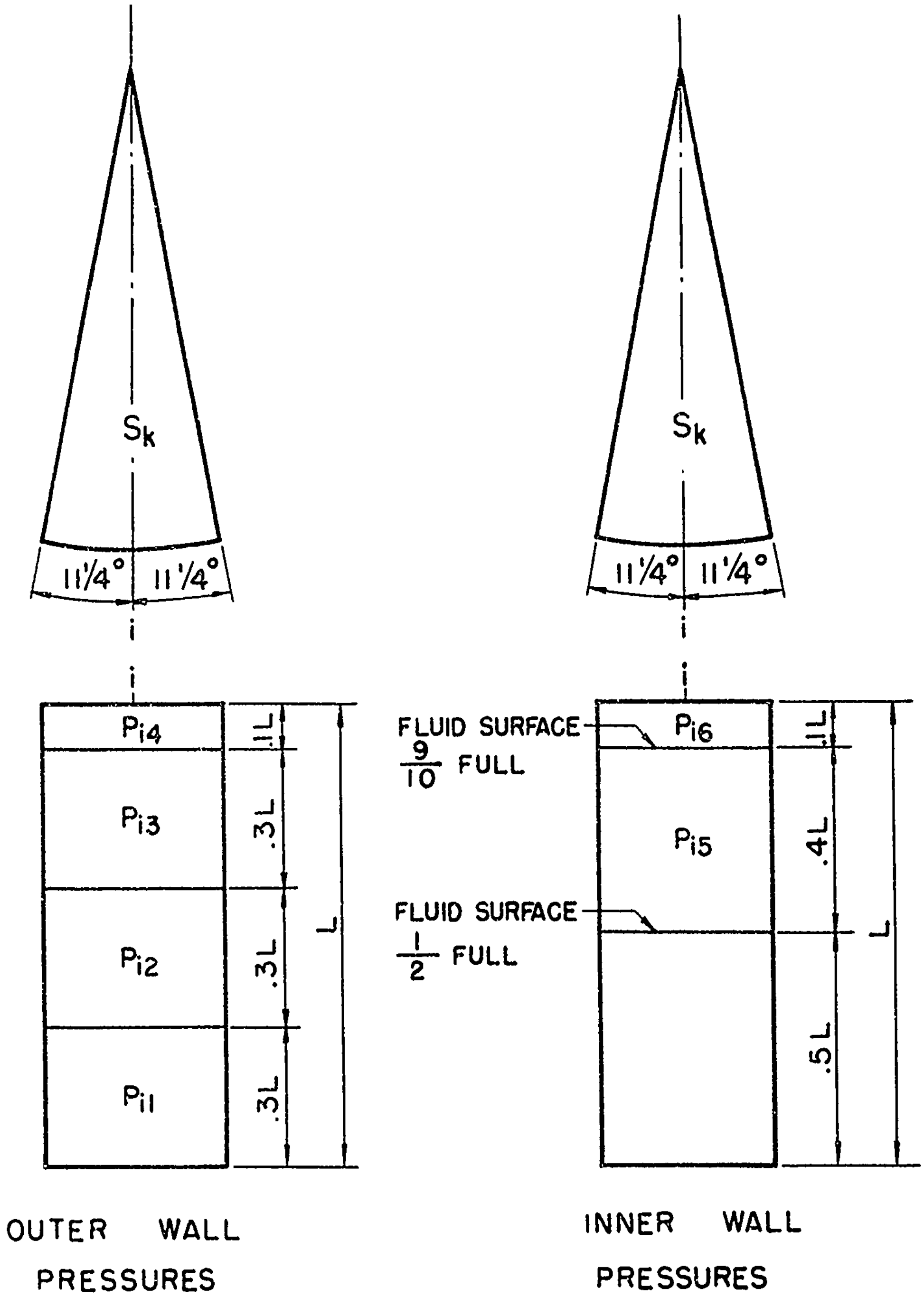

FIG. VII-1 RADIAL WALL PRESSURES ARRANGEMENT AND NOMENCLATURE

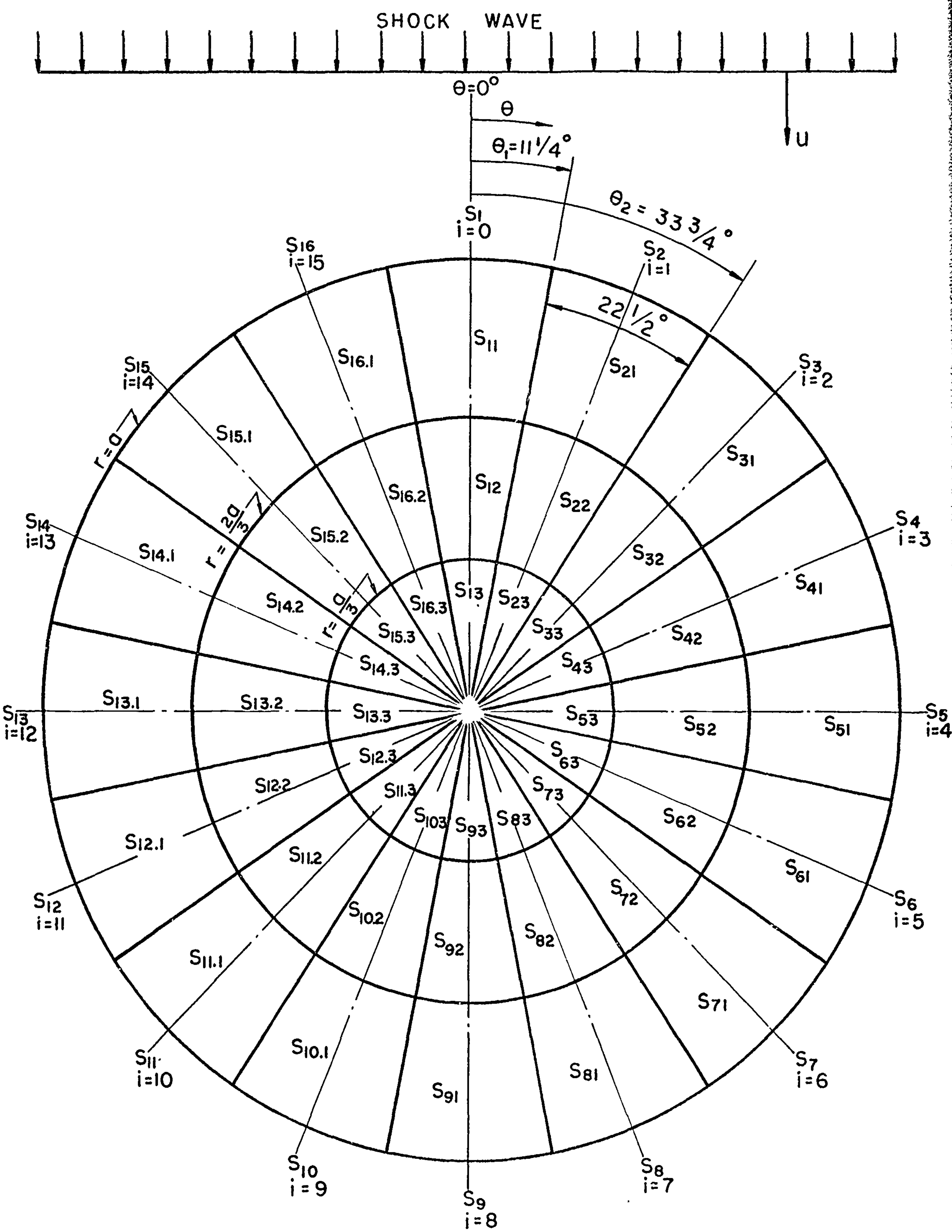

FIG. VII-2 VERTICAL PRESSURE ON FLUID SURFACE ARRANGEMENT AND NOMENCLATURE

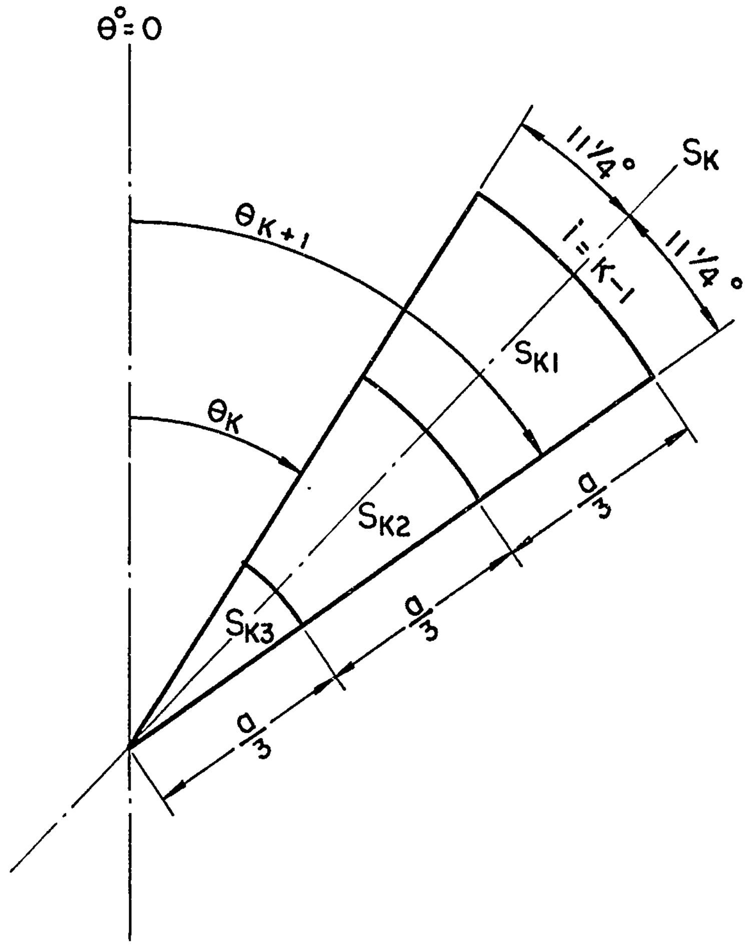

FIG. VII-3 TYPICAL SECTOR S_K
FOR VERTICAL PRESSURES
IN FLUID SURFACE

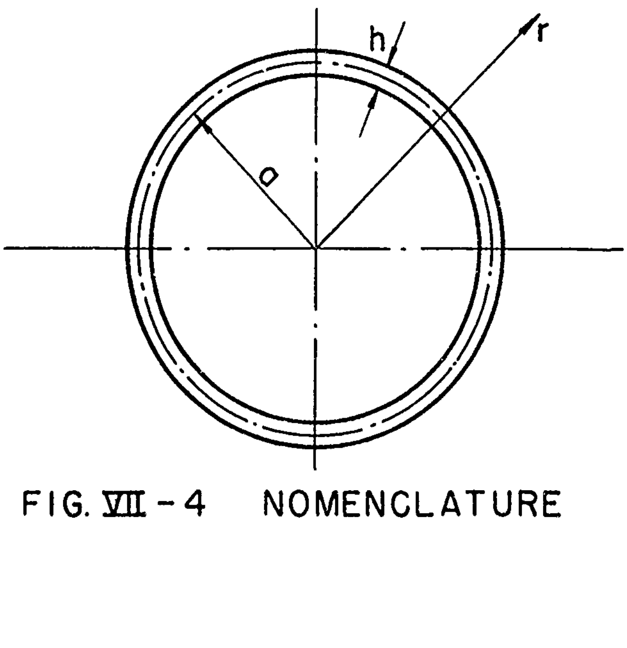

FIG. VII-4 NOMENCLATURE

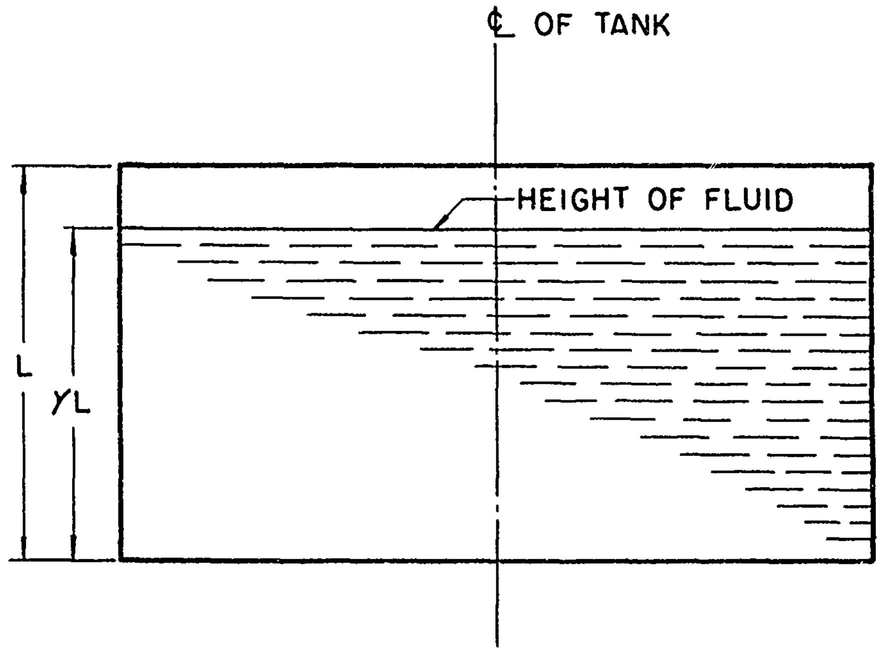

FIG. VII-5

VIII CONCLUSION (Comment on the Purposes and Results of the Mode Analysis)

The logical first step in the analytical study of a new problem concerning the ability of a structure to withstand static or dynamic loads, is the formulation of a theory giving the elastic (usually small deflection) response of the structure. Such a theory will-in general-not be valid up to the stage where the structure collapses or loses its usefulness otherwise; instead, the purpose of this simplest theory is to find out which of the many possible refinements, large deflections, plastic behavior, etc., ought to be included in an analysis to make it valid up to capacity loads.

An obvious example for the situation outlined is the capacity of beams to carry static loads: the conventional (elastic) theory ceases to be valid when the yield point is reached somewhere in the cross section. To obtain a theory valid up to capacity loads plastic effects must be included. It is, however, not always a plastic theory which is required, as may be seen from the example of a slender column. In this well-known case, elastic second order effects must be included to be able to describe the buckling behavior.

Similarly in the present study of the ability of fuel tanks to resist air blast, the mode analysis cannot be expected to be valid up to the end of the usefulness of the structure, because the a-priori unknown lethal damage, possibly tearing of the plates, buckling, sloshing of fluid, necessarily will be preceded by a phase requiring one or more refinements beyond an elastic small deflection theory. The mode analysis presented can however be used--and has been formulated with considerable effort expressly for the purpose of determining which physical effects will

occur requiring a new theory. The sole objective of the mode analysis is therefore to find the end of its own usefulness, occurring at a certain time after a blast wave hits the tank, and disclosing which additional physical effects must be included in a subsequent stage.

The stresses in a typical 100 ft. diameter tank (steel, 9/10 full) were determined by means of this mode analysis, and are presented as sample computation.(1) In addition, less detailed determinations of stresses were made on full, half full and empty full-scale protected and unprotected tanks, and on scaled model tanks ranging from 16"-72" in diameter. These computations indicate in all cases that the mode analysis ceases to be valid after a quite short time,(2) the reason being that the tank wall on the side facing the wave lifts from the support.

The mode analysis assumes that the shell is cantilevered from the base and that the shell is prevented from vertical (and horizontal) motion, an assumption which is initially correct. However, the support of the shell is such that large downward reactions can be carried, while uplift of the shell is prevented only by the dead weight of the shell (and possibly by nominal bolts of small total strength). E.g., in a steel shell of 100 ft. diameter and 40 ft. height, the vertical compressive stress is only 340 lbs./in^2, such that any vertical tensile stress $\sigma_z > 340$ lbs./in^2, induced by the blast will cause uplift. At load intensities of interest, vertical tensile stresses of such small magnitude develop very quickly and the validity of the mode analysis ends due to

(1) See Part (III) of the Final Report

(2) Short in comparison to the decay constant of the shockwave.

uplift for a 100 ft. diameter tank after only a few milliseconds, long before plasticity or large deflections effects can develop.

The theoretical conclusions that uplift will develop has since been verified by tests on mercury filled 16 inch diameter models in the shock tube, and by tests on water filled, full, half-full and empty models of 10 ft. diameter. The tests have demonstrated (See Fig. VIII-1) clearly, that large uplift, exceeding 25 per cent of the height of the model does occur on models prior to lethal damage. This fact leads to the conjecture that appreciable uplift may also occur on full scale tanks.

From the instant of time when the mode analysis becomes invalid, a new type analysis is therefore required; this analysis is complicated by the fact that a part of the shell, (Fig. VIII-2), over some unknown and possibly varying angle 2α will not be vertically supported, while the remainder of the shell, wherever the reactions are downward is still supported by the base. Analytical treatments for this situation are developed in Part II of this Report. These treatments are also made in stages, because the first theory allowing for uplift only requires additional refinements in order to apply up to lethal damage.

In view of this fact that tentative computations and model tests indicate that the mode analysis always ceases to be valid long before lethal damage develops, it appears that the mode analysis need not be repeated at all in a study of effects of various blast waves on tanks of different sizes, [3], but that such a study requires only the application of the methods developed in Part II. It is noted that the work on the

(3) Unless some unconventional type tank is studied.

mode analysis recorded in Part I not only was required to recognize that tanks will lift prior to failure, but that a large part of the derivations are used again in the theories developed in Part II.

FIGURE (VIII-1)

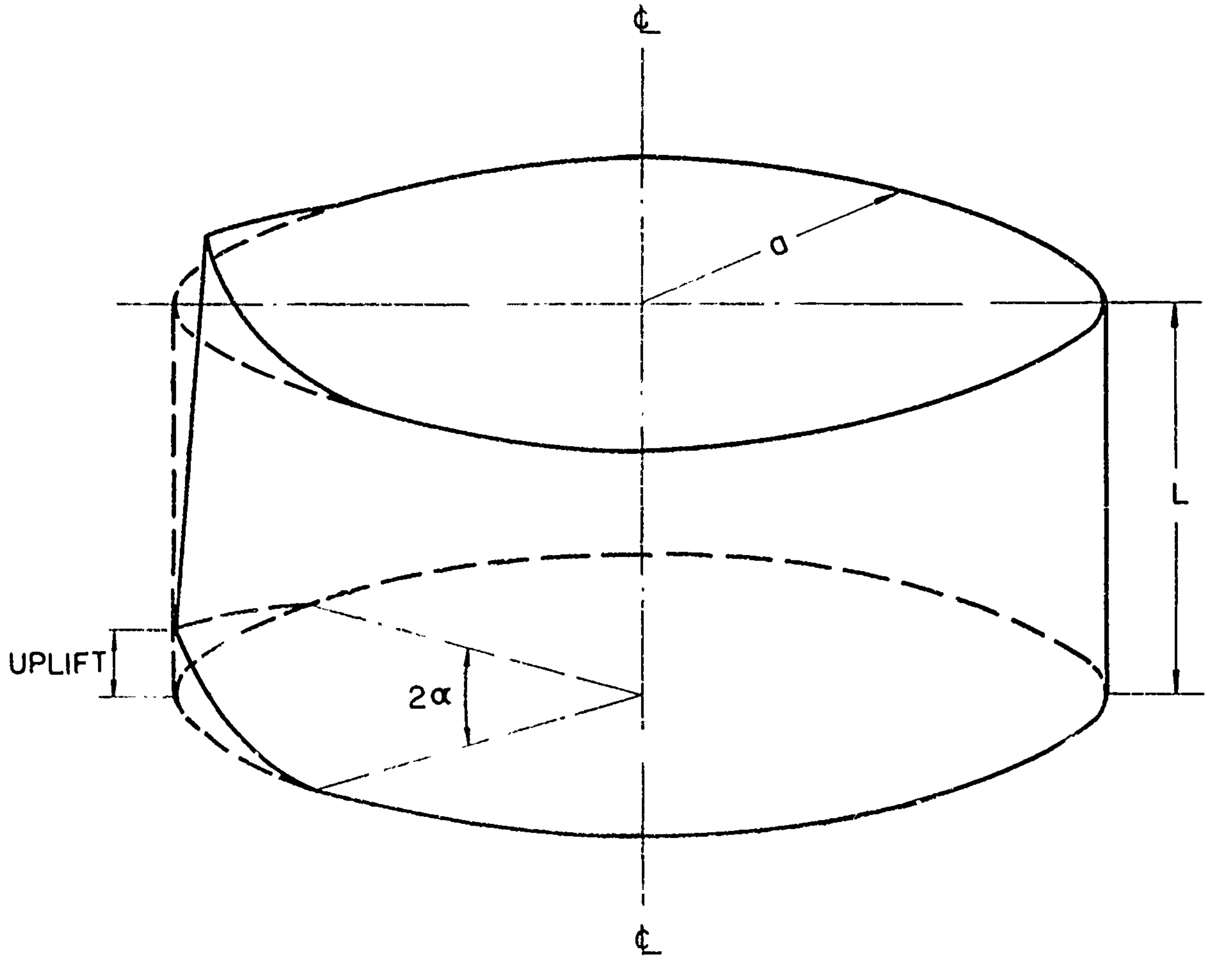

FIG. VIII - 2 UPLIFT IN TANK

Appendix B:Buckling of Empty Cylindrical Tanks

The problem of the static buckling of an empty cylindrical fuel tank subjected to a unform radial pressure producing a uniform compressive hoop stress and strain is considered in this Appendix.[1] The fuel tank is of height L and radius a, and is free at the top and simply supported at the base. The uniform buckling pressure, p_B, is obtained by use of the Rayleigh-Ritz method which leads to a direct solution of the extremum problem arising from the theorem of Stationary Potential Energy.[2] This theorem is given by the following relation:

$$U = \overline{V} + W = \text{stationary} \quad . \qquad \text{(B-1)}$$

where the stationary value of the total potential energy U must be a minimum for stable equilibrium. The quantity $\overline{V}$ represents the strain energy of the structure and W is the potential energy of the external loads.

The general expression for the strain energy $\overline{V}$ in terms of the shell displacements, $u(z, \theta)$, $v(z, \theta)$ and $w(z, \theta)$, is given by Eq. (I-10)-(I-12) of the main paper which are repeated here for convenience:

$$\overline{V} = V_1 + V_2 \qquad \text{(B-2)}$$

---

(1) This problem was considered because of the possibility that tanks might buckle due to the hoop stress produced by the blast loads. It was found subsequently, that in the case of interest, uplift of the tanks occurs prior to any possibility of buckling so that the analysis reported here was not utilized.

(2) "Buckling Strength of Metal Structures" by Friedrich Bleich, Edited by Hans H. Bleich, McGraw-Hill Book Company, 1952, New York, Pg 70 ff.

where the first term V_1 is the membrane strain energy:

$$V_1 = \frac{E}{2(1-\nu^2)} \frac{h}{a} \int_0^{2\pi}\int_0^L \left[a^2 u_z^2 + (v_\theta - w)^2 + 2a\nu u_z (v_\theta - w) + \left(\frac{1-\nu}{2}\right)(u_\theta + a v_z)^2\right] dz d\theta \qquad \text{(B-3)}$$

and the second term V_2 represents the strain energy of bending, and coupling terms between the membrane and bending strains:

$$V_2 = \frac{E}{24(1-\nu^2)} \frac{h^3}{a^3} \int_0^{2\pi}\int_0^L \left[a^4 w_{zz}^2 + (w_{\theta\theta} + w)^2 + \left(\frac{1-\nu}{2}\right)(a w_{z\theta} - u_\theta)^2 + \frac{3(1-\nu)}{2} a^2 (v_z + w_{z\theta})^2 + 2a^2 \nu w_{zz}(w_{\theta\theta} + v_\theta) + 2a^3 u_z w_{zz}\right] dz d\theta \quad . \qquad \text{(B-4)}$$

The expression for W, the potential energy of the external uniform pressure p_B, is given by[(3)]

$$W = \frac{p_B}{2} \int_0^L\int_0^{2\pi} \left[w^2 - w_\theta^2 - u_\theta^2 - 2awu_z\right] d\theta\, dz - \frac{p_B a}{2} \left[\int_0^{2\pi} \nu u_0 \right]_{z = L} d\theta \; . \qquad \text{(B-5)}$$

Using the Rayleigh-Ritz method, an appropriate set of coordinate functions with arbitrary coefficients, which satisfy the boundary conditions on the deflections u, v, and w of the shell is assumed. Substituting these expressions into Eq. (B-1)-(B-5), the total potential energy U is obtained as a function of the arbitrary coefficients. For static equilibrium, the condition that U be a minimum leads to an ordinary maximum-minimum problem on the arbitrary constants and hence to a set of homogeneous

---

(3) "Dynamic Response of Cylindrical Tanks" by F. L. DiMaggio, Armed Forces Special Weapons Project, Contract DA-29-044-XZ-54, AFSWP No. 1075, May 1958, Pg. 21, Eq. (6).

linear algebraic equations on these constants. For non zero solutions, the determinant of the set of homogeneous equations must be set equal to zero, thus leading to an equation on the critical pressure p_{nB} corresponding to a buckling mode defined by the number "n" of circumferential waves in the shell displacements. Buckling determinants will be developed for both the three constant and the five constant approximations considered in Sections (I)-(VIII).

(a) Three Constant Approximation

Let the displacements of the shell (Fig. I-1) be given by the coordinate functions

$$u(z, \theta) = U \frac{z}{a} \cos n\theta \tag{B-6}$$

$$v(z, \theta) = V \frac{z}{a} \sin n\theta \tag{B-7}$$

$$w(z, \theta) = W \frac{z}{a} \cos n\theta \tag{B-8}$$

where U, V and W are undetermined constant coefficients. Substituting into Eq. (B-2)-(B-5), the total strain energy V is given by

$$V = \frac{Eh\pi}{2(1-\nu^2)} \xi \left\{ U^2 + (nV - W)^2 \frac{\xi^2}{3} + \nu U(nV - W)\xi + \left(\frac{1-\nu}{2}\right)\left(V^2 + \frac{n^2\xi^2}{3} U^2 - n\xi\, UV\right) \right\}$$

$$+ \frac{Eh\pi}{24(1-\nu^2)} \frac{h^2}{a^2} \left\{ W^2 \left[\frac{(1-n^2)^2}{3} \xi^3 + 2(1-\nu)n^2\xi\right] + \right.$$

$$+ \left[\left(\frac{1-\nu}{2}\right)\frac{n^2\xi^3}{3}\right] U^2 + \left[\frac{3(1-\nu)}{2} \xi\right] V^2 -$$

$$\left. - \left[\left(\frac{1-\nu}{2}\right)n^2\xi^2\right] UW - \Big[3(1-\nu)\xi n\Big] VW \right\} \tag{B-9}$$

and the potential energy of the external pressure W becomes:

$$W = \frac{p_{nB}\pi L}{2} \left[\frac{(1-n^2)}{3}\xi^2\, W^2 - \frac{n^2\xi^2}{3} U^2 - \xi WU + n\xi VU\right], \tag{B-10}$$

The condition of minimum total potential energy

$$\frac{\partial}{\partial C_i} \left[\bar{V} + W\right] = 0 \tag{B-11}$$

where C_i successively takes the values U, V, W leads to the following set of three homogeneous linear equations in the unknowns U, V and W:

$$U\left[2+(1-\nu)\frac{n^2\xi^2}{3}(1+k)-\frac{2n^2\xi^2}{3}M_B\right]+V\left[n\xi\left(\frac{3\nu-1}{2}\right)+M_Bn\xi\right]+W\left[-\nu\xi-n^2\xi\left(\frac{1-\nu}{2}\right)k-M_B\xi\right]=0$$

(B-12)

$$U\left[n\xi\left(\frac{3\nu-1}{2}\right)+M_Bn\xi\right]+V\left[\frac{2n^2\xi^2}{3}+(1-\nu)+3k(1-\nu)\right]+W\left[\frac{2n\xi^2}{3}-3(1-\nu)nk\right]=0$$

(B-13)

$$U\left[-\nu\xi-n^2\xi\left(\frac{1-\nu}{2}\right)k-M_B\xi\right]+V\left[\frac{2n\xi^2}{3}-3(1-\nu)nk\right]+$$

$$+W\left[\frac{2\xi^2}{3}+k\left\{\frac{2(1-n^2)^2\xi^2}{3}+4(1-\nu)n^2\right\}+\frac{2}{3}(1-n^2)\xi^2M_B\right]=0$$

(B-14)

where

$$M_{nB}=\frac{p_{nB}a(1-\nu^2)}{Eh} \tag{B-15}$$

and $k=\frac{h^2}{12a^2}$. Non zero solutions of Eq. (B-12)-(B-14) require that the determinant of the system vanishes, thus leading to the determenantal buckling equation shown on Page 165.

The lowest root M_{nB} of Eq. (B-16) defines the critical buckling pressure p_{nB} in the mode denoted by n. The buckling stress σ_{nB} can then be computed from the relation

$$\sigma_{nB}=p_{nB}\frac{a}{h}\quad. \tag{B-17}$$

It should be noted that for very thin shells in which $\frac{h}{a}$ is very small, the constant k can be set equal to zero in Eq. (B-16) as in the vibration problem of Section (I).

$$\begin{vmatrix} 2n^2 M_{nB} - \frac{6}{\xi^2} - (1-\nu)n^2(1+k) & -\frac{3n(3\nu-1)}{2\xi} - \frac{3n}{\xi} M_{nB} & \frac{3}{\xi}(\nu + M_{nB}) + \frac{3kn^2(1-\nu)}{2\xi} \\ -\frac{3n(3\nu-1)}{2\xi} - \frac{3n}{\xi} M_{nB} & -2n^2 - \frac{3(1-\nu)(1+3k)}{\xi^2} & +2n + \frac{9kn(1-\nu)}{\xi^2} \\ +\frac{3}{\xi}(\nu + M_{nB}) + \frac{3kn^2(1-\nu)}{2\xi} & +2n + \frac{9kn(1-\nu)}{\xi^2} & -2(1-n^2)M_{nB} - 2 - \frac{6k}{\xi^2}\left\{(1-n^2)^2\frac{\xi^2}{3} + 2(1-\nu)n^2\right\} \end{vmatrix} = 0$$

$n \neq 0$

Equation B-16

Buckling Load Determinant
Membrane + Bending Effects
3 Constant Approximation
Empty Shell

Figure (B-1) shows a curve of the buckling load number M_{nB} plotted against the mode number n for an unprotected steel tank (no concrete shielding). For the lower modes $1 \leq n \leq 7$, the shell acts as a membrane while for the higher modes $n > 18$, the bending strains predominate and control the critical buckling loads.

As an example, the following table shows the static hoop stresses for n = 2, 6, 16 at which a steel tank of 100 ft. diameter, 40 ft. height and 1/2 inch wall thickness would buckle. The smallest hoop stress is required for n = 16.

n	2	6	16
σ_{nB}(psi)	210,000	17,840	720

Figure (B-2) shows the similar curve for a protected steel tank (18 inch concrete protection). For protected steel tanks, the buckling pressures p_{nB} are of course very much higher; in addition the minimum occurs for a much smaller value of n.

(b) Five Constant Approximation.

Let the displacements of the shell be given by the more general coordinate functions

$$u(z,\theta) = \left[U \frac{z}{a} + X \left(\frac{z^2}{a^2} - \frac{3Lz}{4a^2} \right) \right] \cos n\theta \qquad \text{(B-18)}$$

$$v(z, \theta) = V \frac{z}{a} \sin n\theta \qquad \text{(B-19)}$$

$$w(z, \theta) = \left[Y + W \left(\frac{z}{a} - \frac{L}{2a} \right) \right] \cos n\theta \qquad \text{(B-20)}$$

where U, V, W, X and Y are undetermined constant coefficients.

Proceeding exactly as in Part (a) of this Appendix, the determenantal buckling equation shown on Page 168 is obtained.

The lowest root M_{nB} of Eq. (B-21) defines the critical buckling pressure p_{nB} in the mode n.

In sample computations, it was found that the buckling loads are well approximated by the three-constant approximations and that it is unnecessary to use the fifth order determinant, Eq. (B-21). This can be seen from Fig. (B-2).

$$
\begin{vmatrix}
2n^2M_{nB}-\frac{6}{\xi^2}-(1-\nu)n^2(1+k) & -\frac{3n(3\nu-1)}{2\xi}-\frac{3n}{\xi}M_{nB} & \frac{3kn^2(1-\nu)}{2\xi} & -\frac{3}{2\xi} & \frac{6}{\xi^2}(\nu+M_{nB}) \\
-\frac{3n(3\nu-1)}{2\xi}-\frac{3n}{\xi}M_{nB} & -2n^2-\frac{3(1-\nu)}{\xi^2}(1+3k) & \frac{n}{2}+\frac{9kn(1-\nu)}{\xi^2} & -\frac{n(1+13\nu)}{8}-\frac{3}{4}nM_{nB} & \frac{3n}{\xi} \\
\frac{3kn^2(1-\nu)}{2\xi} & \frac{n}{2}+\frac{9kn(1-\nu)}{\xi^2} & -\frac{1}{2}(1-n^2)M_{nB}-\frac{1}{2}-k\left\{\frac{(1-n^2)^2}{2}+\frac{12(1-\nu)n^2}{\xi^2}\right\} & -\frac{k(1-\nu)n^2}{8}+\nu+M_{nB} & 0 \\
-\frac{3}{2\xi} & -\frac{n(1+13\nu)}{8}-\frac{3}{4}nM_{nB} & -\frac{k(1-\nu)n^2}{8}+\nu+M_{nB} & \frac{3}{40}n^2\xi^2M_{nB}-\frac{19}{8}-\frac{3(1-\nu)n^2\xi^2}{80}(1+k) & \frac{3}{2\xi}(\nu+M_{nB}) \\
\frac{6}{\xi^2}(\nu+M_{nB}) & \frac{3n}{\xi} & 0 & \frac{3}{2\xi}(\nu+M_{nB}) & \frac{6}{\xi^2}\left\{-M_{nB}(1-n^2)-1-k(1-n^2)^2\right\}
\end{vmatrix} = 0
$$

Equation B-21

$n \neq 0$

Buckling Load Determinant
Membrane + Bending Effects
5 Constant Approximation
Empty Shell

(c) Dynamic Buckling

The term "dynamic buckling" is only another way of saying that the structure will show a transient response. This transient response can be found again from the appropriate equations of motion using the modes of free vibration as generalized coordinates. In the presence of a hoop stress σ which is a substantial percentage of the buckling stress σ_{nB}, or which exceeds σ_{nB}, Eq. (VII-10) the equation of motion for the generalized coordinate q_n becomes

$$\ddot{q}_n + \omega_n^2 \left(1 - \frac{\sigma}{\sigma_{nB}} \right) q_n = \frac{Q_n}{\overline{M}_i} \quad . \qquad \text{(B-22)}$$

If σ/σ_{nB} is small, such that it can be neglected, the original equation of motion, Eq. (VII-10) is again obtained. It is noted that Eq. (B-22) is valid for both positive or negative values of $1 - \dfrac{\sigma}{\sigma_{nB}}$.

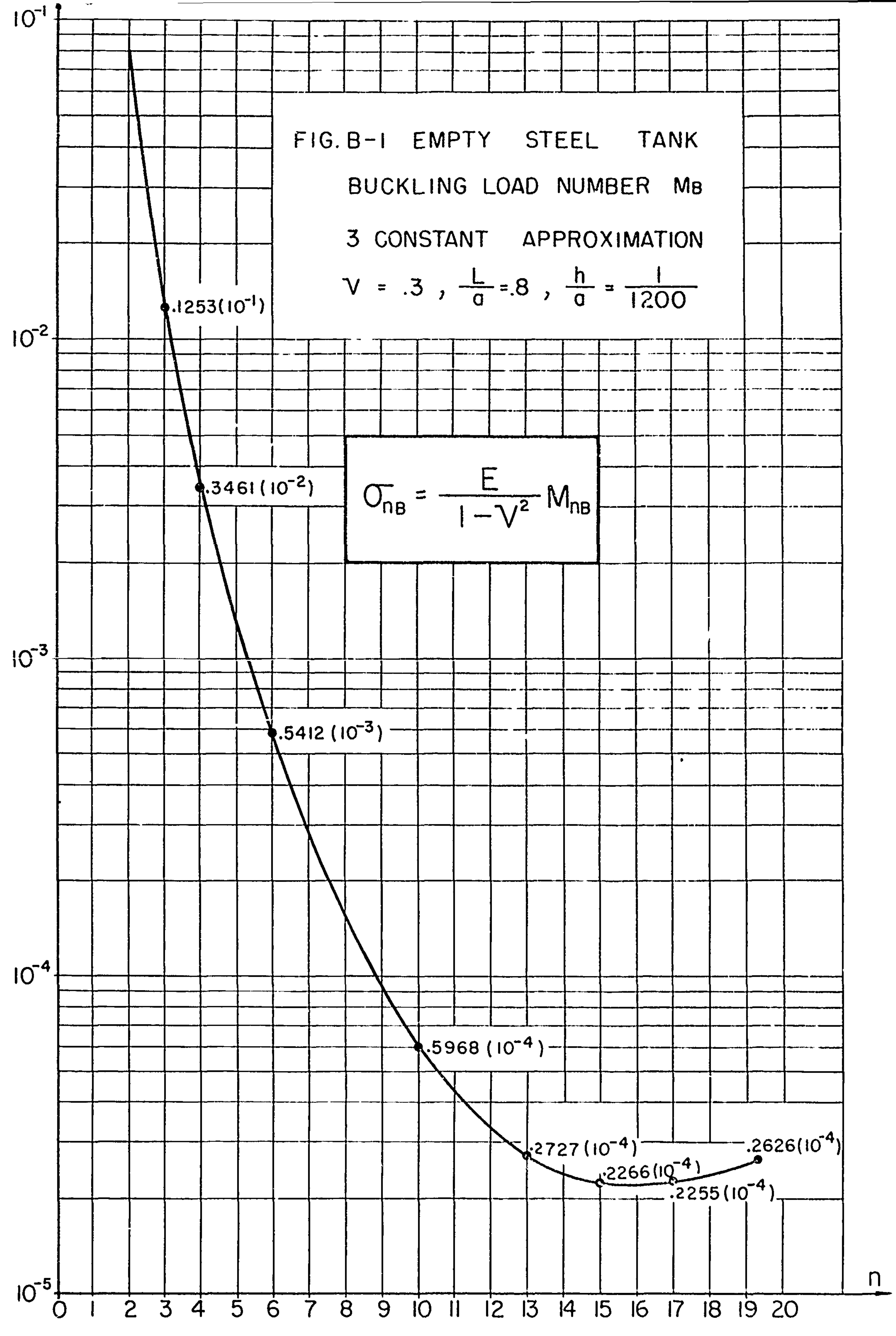

FIG. B-1 EMPTY STEEL TANK
BUCKLING LOAD NUMBER MB
3 CONSTANT APPROXIMATION
ν = .3 , L/a = .8 , h/a = 1/1200
σnB = E/(1−ν²) MnB
.1253(10⁻¹)
.3461(10⁻²)
.5412(10⁻³)
.5968(10⁻⁴)
.2727(10⁻⁴)
.2266(10⁻⁴)
.2255(10⁻⁴)
.2626(10⁻⁴)
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
n
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

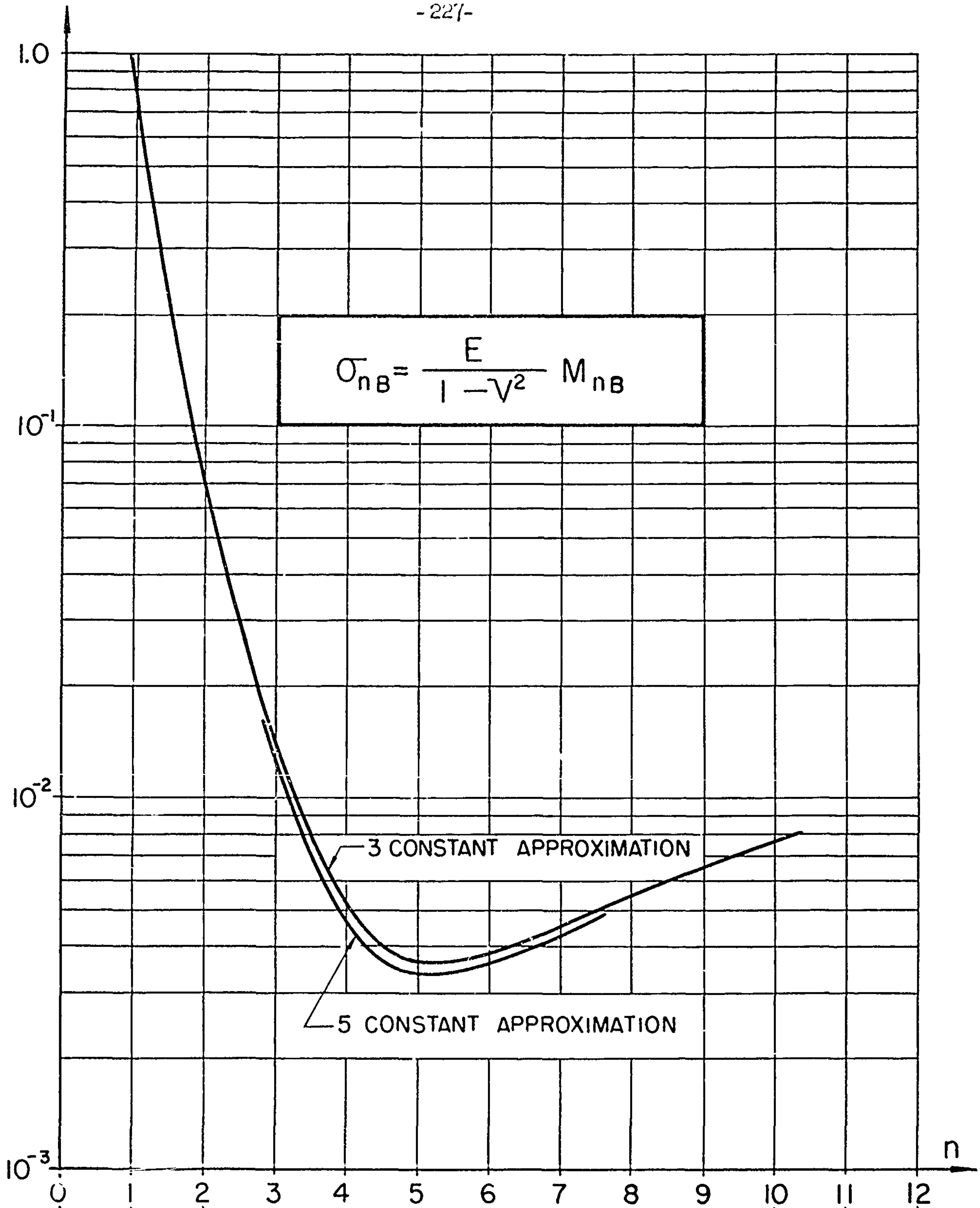

FIG. B-2 EMPTY CONCRETE TANK

BUCKLING LOAD NUMBER M_{nB}

3 & 5 CONSTANT APPROXIMATION

$\nu = .3$, $\frac{L}{a} = .8$, $\frac{b}{a} = \frac{18.5}{600}$